Enhancing Credit Default Swap Valuation With Meshfree Methods

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April 2011

Abstract

In this paper, we apply the meshfree radial basis function (RBF) interpolation to numerically approximate zero-coupon bond prices and survival probabilities in order to price credit default swap (CDS) contracts. We assume that the interest rate follows a Cox-Ingersoll-Ross process while the default intensity is described by the Exponential-Vasicek model. Several numerical experiments are conducted to evaluate the approximations by the RBF interpolation for one- and two-factor models. The results are compared with those estimated by the finite difference method (FDM). We find that the RBF interpolation achieves more accurate and computationally efficient results than the FDM. Our results also suggest that the correlation between factors does not have a significant impact on CDS spreads.

Keywords: Radial Basis Function Interpolation, Finite Difference Methods.

JEL Codes: C63, C65, G12, G13

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*We are grateful to the editor and two anonymous referees for helpful comments which have substantially improved the exposition of the paper.
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1 Introduction

In this paper, we adopt meshfree methods as an alternative to mesh-based techniques in pricing credit derivatives. In particular, we approximate zero-coupon bond prices and survival probabilities in a reduced form model using the radial basis function (RBF) interpolation in an effort to accurately and efficiently evaluate credit default swap (CDS) contracts.

CDS contracts have been the most popular single-name credit derivatives in the market in recent years. They are traded to cover risk exposure and speculate. Consequently, these instruments contain important information on the dynamics of the term structure of default probability that help us understand and quantify the level of credit risk affecting individual firms and the whole economy. Brigo and Mercurio (2006) highlight that an accurate estimation of the credit risk implicit in CDS spreads becomes essential not only in the valuation of CDS contracts themselves but also in the pricing and risk management of more complex credit derivatives with similar risk exposure.

Reduced form models, which assume default as an exogenous random event triggered at any time, are one of the main approaches in credit risk management literature\(^1\). They are considered a suitable approach for modelling credit spreads and calibrating market data for CDS contracts (Brigo and Mercurio (2006)). Most literature adopting these models assumes multi-factor stochastic processes with closed-form solutions. Recent examples include Duffie and Singleton (1999), Bakshi et al. (2006) and Jacobs and Li (2008) who analyze defaultable corporate bond prices of individual firms. Longstaff et al. (2005), Ueno and Baba (2006), Chen et al. (2008a) and Chen et al. (2008b) study corporate CDS contracts, while Zhang (2003) and Carr and Wu (2007) focus on sovereign CDS contracts. Onorato and Altman (2005) adopt an integrated model for credit risk based upon a reduced pricing framework.

Reduced form models also adopt stochastic processes with no analytical solution. Recent examples in this strand of literature include Realdon (2007) and Pan

\(^1\)The other main approach utilizes structural models where the default time is the first instant when the value of a firm hits a deterministic or stochastic barrier. See, for example, Merton (1974), Black and Cox (1976), Giesecke (2006), and Kreinin and Nagi (2008).
and Singleton (2008), who model sovereign CDS contracts assuming that default intensity follows a lognormal process in one- and two-factor models. They also show that the inclusion of a second factor is statistically and economically significant and can better capture the variation in CDS spreads. Jobst and Zenios (2005) examine the valuation of credit portfolios that are subject to interest rate and credit risks. As there is no closed-form solution for these models, numerical methods are needed.

The valuation of CDS contracts involves two main components: first, a discount factor which can be modelled as the zero-coupon bond price; and second, the survival probability. In a stochastic setting, the value of these components can be computed as solutions of partial differential equations (PDE) describing the dynamics of the interest rate and the default intensity. A numerical approach is required if there is no closed-form solution for the underlying model or if a closed-form solution is based on restrictive assumptions.

Numerical approximation methods have become an important tool in financial engineering as a result of increasingly complex derivative pricing models. Nevertheless, most practitioners employ mesh-based methods such as the finite difference method (FDM). Although this technique has been very popular in the literature, it faces difficulties in complex problems associated with the use of a regular and fixed grid (Duffy (2006)). Most of these drawbacks are overcome by the meshfree approximation methods, which provide outstanding results in terms of accuracy and computational efficiency. The meshfree methods use a set of points scattered on the domain of the problem instead of a grid. Hence, they are adaptive, versatile and robust tools to deal with complex geometries, irregular discretization and high-dimensional problems (Koc et al. (2003), Liu (2003), and Fasshauer (2007)).

As main contribution of this paper, we apply the meshfree RBF interpolation method to obtain an accurate and efficient approximation of the dynamics of zero-coupon bond prices and survival probabilities in a reduced form model setting. The numerical results are subsequently used in valuing CDS spreads for one- and two-factor models and, in the case of two-factor models, with and without correlation between factors. To the best of our knowledge, this is the first study that adopts the RBF interpolation for credit derivatives valuation, contributing to a recent but
In particular, we employ the Cox-Ingersoll-Ross (CIR) process (Cox et al. (1985)), a popular choice in the literature, for modelling the interest rate. At the same time, we assume that the default intensity follows the Exponential-Vasicek (EV) model (Brigo and Mercurio (2006)), whose underlying process follows a lognormal distribution. Brigo et al. (2009) highlight the appealing features of this process, including mean reversion, strictly positive values, positive skewness and fat tails, although the model does not have a closed-form solution.

Our second contribution is that we carry out comparisons between the numerical results obtained by the RBF interpolation and the FDM. For zero-coupon bond prices, we use the analytical solution as benchmark and compare approximations by these two methods. In the case of survival probabilities, as there is no analytical solution, we make direct comparison between the numerical approaches. We also present the CDS spreads thus obtained.

Our results show that the RBF interpolation outperforms the FDM in terms of accuracy and efficiency, even when the number of points is small. The potential gain of using the RBF interpolation grows with the dimension of the problem and we find that the CDS pricing differences are larger when we assume two-factor models for interest rates and default intensities. Our results also indicate that the correlation between factors does not have a significant impact on CDS spreads.

The remainder of the paper is organized as follows. Section 2 defines the CDS payoff and outlines the valuation of CDS contracts in a reduced form setting. In Section 3, we employ the RBF interpolation to approximate the solution for zero-coupon bond prices and survival probabilities in a multi-factor model. Section 4 compares and analyzes results from numerical experiments. Finally, Section 5 concludes.

2 Credit Default Swap

A CDS is a private contract between a protection buyer (“long side”) and protection seller (“short side”). In this agreement, the CDS seller ensures protection to the buyer against a credit default event (e.g. failure to pay, bankruptcy, etc) of a reference
obligation (e.g. loan, bond, etc) issued by a reference entity (e.g. a company or government, etc).

We consider a CDS contract in the time interval \([0, T_b]\) with spread \(R\). Let \(\Pi_{CDS_{0,b}}\) denote the expected payoff of the \(CDS_{0,b}(R)\) at time 0. For the protection seller, \(\Pi_{CDS_{0,b}}\) is computed as the expected difference between the premium leg and the protection leg. The premium leg has two components: First, the regular payments that a protection buyer makes at times \(T_1, \ldots, T_b\) to a protection seller until either the credit obligation defaults at time \(\tau \in (0, T_b)\) or the CDS contract matures at time \(T_b\) without defaulting. Second, the accrued amount between the most recent payment date \(T_{\beta(\tau)-1}\) and the time \(\tau\) in the case of default. In contrast, the protection leg consists of the contingent payment that a CDS seller makes to the buyer if the credit obligation under reference defaults at time \(\tau \in (0, T_b)\). This cashflow is 0 if defaults does not happen. Following Brigo and Mercurio (2006), we define \(\Pi_{CDS_{0,b}}\) as

\[
\Pi_{CDS_{0,b}} = \sum_{i=1}^{b} \mathbb{E} \left[ D(0, T_i) \alpha_i R \mathbf{1}_{\{\tau \geq T_i\}} \right] + \mathbb{E} \left[ D(0, \tau) \left( \tau - T_{\beta(\tau)-1} \right) R \mathbf{1}_{\{0 < \tau < T_b\}} \right] \\
- \mathbb{E} \left[ \mathbf{1}_{\{0 < \tau \leq T_b\}} D(0, \tau) L_{GD} \right],
\]

where \(D(0, \cdot)\) is the discount factor at time 0 for maturity \((\cdot)\), \(\alpha_i\) is the fraction in years between \(T_{i-1}\) and \(T_i\), \(\mathbf{1}_{\{\cdot\}}\) is an indicator function (with 1 if condition \((\cdot)\) is satisfied and 0 otherwise), and \(L_{GD}\) is the loss given default computed as \(1 - R_{EC}\), where \(R_{EC}\) is the recovery rate of the underlying credit obligation.

In order to obtain a final expression for \(\Pi_{CDS_{0,b}}\), we assume a stochastic setting for the short interest rate \(r\). The default time \(\tau\) is modelled as the first jump of a Cox process with stochastic default intensity \(\lambda\). We also consider that \(D(0, \cdot)\) and \(\tau\) are independent, therefore \(\mathbb{E} \left[ D(0, \cdot) \mathbf{1}_{\{\cdot\}} \right] = \mathbb{E} \left[ D(0, \cdot) \right] \mathbb{E} \left[ \mathbf{1}_{\{\cdot\}} \right]\). This assumption is not restrictive as discussed in Brigo and Alfonsi (2005). Under these assumptions,
Brigo and Mercurio (2006) show that the expected payoff $\Pi_{CDS_{0,b}}$ is

$$
\Pi_{CDS_{0,b}} := R \left[ \sum_{i=1}^{b} P(0, T_i) \alpha_i Q(\tau \geq T_i) - \int_0^{T_b} P(0, t) \left( t - T_{\beta(t)} - 1 \right) d_t Q(\tau \geq t) \right] + L_{GD} \left[ \int_0^{T_b} P(0, t) d_t Q(\tau \geq t) \right],
$$

(2)

with

$$
P(0, \cdot) = \mathbb{E}[D(0, \cdot)] = \mathbb{E}\left[ \exp^{-\int_0^t r_s ds} \right],
$$

(3)

$$
Q(\tau \geq \cdot) = \mathbb{E}\left[ \exp^{-\int_0^t \lambda_s ds} \right],
$$

(4)

where $P(0, \cdot)$ is the zero-coupon bond price at time 0 for maturity $(\cdot)$, $Q(\tau \geq \cdot)$ is the probability at time 0 of surviving to a future time $(\cdot)$, and $r_s$ and $\lambda_s$ are the short interest rate and default intensity at time $s$, respectively. In order to find the fair spread $R$, we set $\Pi_{CDS_{0,b}} = 0$ in equation (2) such that

$$
R = \frac{-L_{GD} \left[ \int_0^{T_b} P(0, t) d_t Q(\tau \geq t) \right]}{-\int_0^{T_b} P(0, t) \left( t - T_{\beta(t)} - 1 \right) d_t Q(\tau \geq t) + \sum_{i=1}^{b} P(0, T_i) \alpha_i Q(\tau \geq T_i)}.
$$

(5)

Hence, we need to model the dynamics for the interest rate and the default intensity.

### 2.1 CIR Interest Rate Model

Let the short interest rate $r_s$ follow a multi-factor CIR model. Define $r_s$ as

$$
r_s = \sum_{i=1}^{N} x_{i,s}, \quad i = 1, \ldots, N,
$$

(6)

where $x_{i,s}$ is the latent factor $i$ at time $s$. Following Brigo and Mercurio (2006), for a suitable choice of the market price of risk, the factor $x_{i,s}$ under the risk-neutral measure $Q$ follows the process

$$
dx_{i,s} = \kappa_i (\theta_i - x_{i,s}) ds + \sigma_i \sqrt{x_{i,s}} dW_{i,s}, \quad i = 1, \ldots, N,
$$

(7)
with correlation \( \rho_{i,j} = \mathbb{E}[dW_{i,s}dW_{j,s}], \quad i,j = 1, \ldots, N \). The parameters \( \kappa_i, \theta_i \) and \( \sigma_i \) are the speed of reversion, the long term mean level and the instantaneous volatility, respectively, and \( dW \) is a \( \mathcal{Q} \)-Brownian motion.

Replacing equation (6) in (3), we obtain the following expression for the zero-coupon bond price

\[
P(x; 0, t) = \mathbb{E} \left[ \exp - \int_0^t \sum_{i=1}^N x_{i,s} ds \right],
\]

where \( x \) is an \( N \)-dimensional vector of latent factors \( x = \{ x_1, \ldots, x_N \} \).

Equation (8) has a closed-form solution. Nevertheless, we also compute the value of \( P(x; 0, t) \) by numerical approximation. The analytical solution is used as benchmark to evaluate the performance of the RBF interpolation and the FDM later in Section 4. From equations (7) and (8), the zero-coupon bond price by the Feynman-Kac theorem is given by the solution of the PDE

\[
\frac{\partial P}{\partial t} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \rho_{i,j} \sigma_i \sigma_j \sqrt{x_i} \sqrt{x_j} \frac{\partial^2 P}{\partial x_i \partial x_j} + \sum_{i=1}^N \kappa_i (\theta_i - x_i) \frac{\partial P}{\partial x_i} - \left[ \sum_{i=1}^N x_i \right] P = 0,
\]

\[
0 < x_i < \infty, \quad 0 < x_j < \infty, \quad i, j = 1, \ldots, N, \quad 0 < t \leq T,
\]

with the following initial and boundary conditions

\[
P(x; 0, t) = 1, \quad 0 \leq x_i < \infty, \quad i = 1, \ldots, N, \quad t = 0,
\]

\[
\frac{\partial P}{\partial t} + \kappa_i \theta_i \frac{\partial P}{\partial x_i} = 0, \quad x_i = 0, \quad i = 1, \ldots, N, \quad 0 < t \leq T,
\]

\[
P(x; 0, t) = 0, \quad x_i \to \infty, \quad i = 1, \ldots, N, \quad 0 < t \leq T.
\]

The boundary condition when \( x_i = 0 \) is by itself a PDE. The solution for \( P(x; 0, t) \) is obtained by numerical approximation of this latter system.

### 2.2 EV Default Intensity Model

We assume that the default intensity \( \lambda_s \) follows a multi-factor EV process (see Brigo and Mercurio (2006)). This is equivalent to the restricted version of the Black and Karasinski (1991) model with constant coefficients over time. Let \( \lambda_s \) under a
multi-factor EV model be defined as

$$\lambda_s = \sum_{i=1}^{M} \exp^{y_{i,s}}, \quad i = 1, \ldots, M,$$  \hspace{1cm} (12)

where $y_{i,s}$ is the latent factor $i$ at time $s$. For a suitable choice of the market price of risk, the factor $y_{i,s}$ under the risk-neutral measure $Q$ follows the process

$$dy_{i,s} = \eta_i (\ln \mu_i - y_{i,s}) \, ds + v_i dW_{i,s}, \quad i = 1, \ldots, M, \hspace{1cm} (13)$$

with correlation $\gamma_{i,j} = \mathbb{E}[dW_{i,s}, dW_{j,s}], \quad i, j = 1, \ldots, M$. The parameter $\ln \mu_i$ is the long term mean level, $\eta_i$ is the speed of reversion at which $y_i$ tends to its long-term value, $v_i$ is the instantaneous volatility, and $dW$ is a $Q$-Brownian motion. Substituting (12) in equation (4), we obtain the survival probability

$$Q(y; \tau \geq t) = \mathbb{E}\left[ \exp^{-\int_0^t \left[ \sum_{i=1}^{M} \exp^{y_{i,s}} \right] ds} \right], \hspace{1cm} (14)$$

where $y$ is an $M$-dimensional vector of latent factors $y = \{y_1, \ldots, y_M\}$.

Equation (14) does not have a closed-form solution and therefore $Q(y; \tau \geq t)$ must be approximated by numerical methods. From equations (13) and (14), the value of the survival probability by the Feynman-Kac theorem is given by the solution of the PDE

$$\frac{\partial Q}{\partial t} + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \gamma_{i,j} v_i v_j \frac{\partial^2 Q}{\partial y_i \partial y_j} + \sum_{i=1}^{M} \eta_i (\ln \mu_i - y_i) \frac{\partial Q}{\partial y_i} - \left[ \sum_{i=1}^{M} \exp^{y_i} \right] Q = 0,$$  \hspace{1cm} (15)

with the following initial and boundary conditions

$$Q(y; \tau \geq t) = 1, \quad -\infty < y_i < \infty, \quad i = 1, \ldots, M, \quad t = 0, \hspace{1cm} (16)$$

$$\frac{\partial Q(y; \tau \geq t)}{\partial y_i} = 0, \quad y_i \to -\infty, \quad i = 1, \ldots, M, \quad 0 < t \leq T, \hspace{1cm} (17)$$

$$Q(y; \tau \geq t) = 0, \quad y_i \to \infty, \quad i = 1, \ldots, M, \quad 0 < t \leq T.$$
3 Meshfree Methods

Meshfree (or meshless) methods are a collection of computational techniques often adopted in dealing with problems requiring accurate, efficient and robust numerical solutions (Fasshauer (2007)). They are based on a set of points called nodes, which are scattered over the domain and boundaries of the problem. As there is no predefined mesh or relationship among the nodes, the methods are easy to implement, especially for multi-dimensional problems. In addition, the allocation of points can be generated automatically so that the implicit cost associated with mesh creation is avoided (Liu (2003)). The accuracy and efficiency of these methods depend on the number of points and their distribution on the space of the problem, both of which can be modified to improve results (Liu (2003) and Liu and Gu (2005)). Hence, the methods overcome the weaknesses of mesh-based methods in terms of discretization, the construction of underlying grids and their regularity conditions (Duffy (2006) and Fasshauer (2007)).

In recent years, meshfree methods have become an interdisciplinary tool for a wide range of fields such as computer graphics, artificial intelligence, neural networks, data mining, signal processing, optimization and nanotechnology (Liu (2003) and Fasshauer (2007)). In finance, the methods have been implemented in option pricing. Hon and Mao (1999), Koc et al. (2003) and Fasshauer et al. (2004) adopt the methods in approximating the Black-Scholes-Merton partial differential equation to value vanilla European and American options. Similarly, Choi and Marcozzi (2001) and Fasshauer et al. (2008) approximate the price of American options on foreign currencies and non-smooth payoffs, respectively. Other studies in option valuation include Hon (2002), Pettersson et al. (2008) and Larsson et al. (2008).

3.1 RBF Interpolation

One of the most popular meshfree methods is the radial basis function (RBF) interpolation. This is a powerful tool for multivariate approximation from scattered data (Iske (2004)). This method uses a set of quasi-random points over the space. It deals with univariate basis functions and the Euclidean norm to reduce a multi-
dimensional problem to a one-dimensional issue (Fasshauer (2007)). In addition, the technique works well with correlation terms without requiring special treatment (Fasshauer et al. (2004) and Duffy (2006)). This feature is of crucial importance for the growing market of multi-dimensional derivative products.

In specific, the method approximates the value of a function as the weighted sum of RBFs evaluated on a set of points, which are quasi-randomly scattered. The weights are found by matching the approximated and observed values of the function on the location nodes called centers. Once we have the interpolation weights we can estimate the value of the function at any point over the space of the problem (for more details see Fasshauer (2006, 2007)).

Following Fasshauer (2007), let us define the set of centers \( Z = [z_1, \ldots, z_K]' \) with \( z_k \in \mathbb{R}^d, \ d \geq 1 \) and the data values \( g_k \in \mathbb{R} \). We assume that \( g_k = f(z_k, t), \ k = 1, \ldots, K \), where \( f \) is an unknown function and \( t \) is the time. We also define \( f(Z, t) \) as a linear combination of \( K \) certain basis functions

\[
f(Z, t) \simeq \sum_{k=1}^{K} \delta_k(t) \varphi(\| Z - z_k \|), \quad k = 1, \ldots, K,
\]

where the coefficients \( \delta_k(t) \) are the unknown weights, \( \varphi(\cdot) \) is the chosen radial basis function and \( \| Z - z_k \| \) is the Euclidean norm. As Fasshauer (2007) shows, equation (18) is in essence a system of linear equations which we must solve to obtain the interpolation coefficients \( \delta_k(t) \). Once these weights are found, we can estimate the value of the function \( f \) at any set of points \( \tilde{Z} = [\tilde{z}_1, \ldots, \tilde{z}_L]' \) with \( \tilde{z}_l \in \mathbb{R}^d \) for \( l = 1, \ldots, L \) and time \( t \) as

\[
f(\tilde{Z}, t) \simeq \sum_{k=1}^{K} \delta_k(t) \varphi(\| \tilde{Z} - z_k \|).
\]

### 3.2 Approximating the Survival Probability

As we have discussed above, the solution for the survival probability in equation (14) is given by the PDE (15) subject to the initial and boundary conditions provided in equations (16) and (17), respectively. Following Koc et al. (2003), we use the
Crank-Nicolson averaging to discretize equation (15) in time such that

\[
\frac{Q(y; \tau \geq t) - Q(y; \tau \geq t + \Delta t)}{\Delta t} + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \gamma_{i,j} v_i v_j \frac{\partial^2 Q(y; \tau \geq t + \Delta t/2)}{\partial y_i \partial y_j} + \sum_{i=1}^{M} \eta_i \left(\ln \mu_i - y_i\right) \frac{\partial}{\partial y_i} \left[\sum_{i=1}^{M} \exp^{y_i} \right] Q \left(y; \tau \geq t + \Delta t/2 \right) = 0, \quad (19)
\]

where \(\Delta t\) is the time step and \(Q \left(y; \tau \geq t + \Delta t/2 \right) = \frac{1}{2} \left[Q(y; \tau \geq t) + Q(y; \tau \geq t + \Delta t)\right].\)

Next, we separate the variables at time \(t\) and \(t + \Delta t\) on each side of the equation, so that equation (19) can be rewritten as

\[
H_+^Q Q(y; \tau \geq t + \Delta t) = H_-^Q Q(y; \tau \geq t), \quad (20)
\]

where \(H_+^Q\) and \(H_-^Q\) are the operators

\[
H_+^Q = 1 - \frac{\Delta t}{2} \left[\frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \gamma_{i,j} v_i v_j \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{M} \eta_i \left(\ln \mu_i - y_i\right) \frac{\partial}{\partial y_i} \left[\sum_{i=1}^{M} \exp^{y_i} \right]\right]
\]

\[
H_-^Q = 1 + \frac{\Delta t}{2} \left[\frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \gamma_{i,j} v_i v_j \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{M} \eta_i \left(\ln \mu_i - y_i\right) \frac{\partial}{\partial y_i} \left[\sum_{i=1}^{M} \exp^{y_i} \right]\right].
\]

Now we replace variable \(Q\) in equation (20) by the linear combination of RBFs

\[
Q(Y, t) \simeq \sum_{k=1}^{K} \delta_k^Q(t) \varphi \left(\|Y - y^k\|\right), \quad k = 1, \ldots, K, \quad (21)
\]

where \(Y\) is the set of \(K\) centers, \(Y = [y_1, \ldots, y_K]^t\), with \(y^k = [y_1^k, \ldots, y_M^k]\) to be evaluated; the coefficients \(\delta_k^Q(t)\) for \(k = 1, \ldots, K\) at time \(t\) are the weights and \(\varphi \left(\| \cdot \|\right)\) is the chosen RBF. Finally, we obtain the system

\[
\sum_{k=1}^{K} \delta_k^Q(t + \Delta t) H_+^Q \varphi \left(\|Y - y^k\|\right) = \sum_{k=1}^{K} \delta_k^Q(t) H_-^Q \varphi \left(\|Y - y^k\|\right). \quad (22)
\]

To obtain the solution \(\delta_k^Q(t + \Delta t)\), we have to solve the linear system (22) iteratively given the values \(\delta_k^Q(t)\) from the previous step. The initial value for \(\delta_k^Q(t)\) is derived from equation (21) and the initial condition (16). The boundary conditions must
be satisfied through the iterative solution of the system.

3.3 Approximating the Zero-Coupon Bond Price

The PDE (9) subject to the initial and boundary conditions in equations (10) and (11) provides the solution for the zero-coupon bond price described in equation (8). We follow the same procedure outlined above to approximate equation (9) by the RBF interpolation approach in space and by the Crank-Nicolson averaging in time, yielding

\[
P(\mathbf{x}; 0, t) - P(\mathbf{x}; 0, t + \Delta t) = \frac{1}{\Delta t} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{ij} \sigma_{ij} \sqrt{x_i} \sqrt{x_j} \frac{\partial^2 P(\mathbf{x}; 0, t + \Delta t)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \kappa_i (\theta_i - x_i) \frac{\partial P(\mathbf{x}; 0, t + \Delta t)}{\partial x_i} - \sum_{i=1}^{N} x_i P(\mathbf{x}; 0, t + \Delta t) = 0,
\]

where \(\Delta t\) is the time step and \(P(\mathbf{x}; 0, t + \frac{\Delta t}{2}) = \frac{1}{2} [P(\mathbf{x}; 0, t) + P(\mathbf{x}; 0, t + \Delta t)]\).

Separating the elements at time \(t\) and \(t + \Delta t\) on each side of the equation and replacing \(P\) by the linear combination of RBFs

\[
\sum_{k=1}^{K} \delta_k^P(t) \varphi(\| \mathbf{X} - \mathbf{x}^k \|), \quad k = 1, \ldots, K,
\]

and evaluating the set of \(K\) centers \(\mathbf{X} = [x_1, \ldots, x_K]'\) with \(\mathbf{x}^k = [x^k_1, \ldots, x^k_N]\), we obtain the linear system

\[
\sum_{k=1}^{K} \delta_k^P(t + \Delta t) H^P_+ \varphi(\| \mathbf{X} - \mathbf{x}^k \|) = \sum_{k=1}^{K} \delta_k^P(t) H^P_- \varphi(\| \mathbf{X} - \mathbf{x}^k \|),
\]

where \(H^P_+\) and \(H^P_-\) are the operators

\[
H^P_+ = 1 - \frac{\Delta t}{2} \left[ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{ij} \sigma_{ij} \sqrt{x_i} \sqrt{x_j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \kappa_i (\theta_i - x_i) \frac{\partial}{\partial x_i} - \sum_{i=1}^{N} x_i \right]
\]

\[
H^P_- = 1 + \frac{\Delta t}{2} \left[ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{ij} \sigma_{ij} \sqrt{x_i} \sqrt{x_j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \kappa_i (\theta_i - x_i) \frac{\partial}{\partial x_i} - \sum_{i=1}^{N} x_i \right].
\]
The coefficients $\delta^P_k (t)$ for $k = 1, \ldots, K$ at time $t$ are the weights and $\varphi (\| \cdot \|)$ is the chosen RBF. To find the weights $\delta^P_k (t + \Delta t)$, we have to solve the linear system (25) iteratively given the values $\delta^P_k (t)$ from the previous step. The initial value for $\delta^P_k (t)$ is obtained from equation (24) and the initial condition (10). The boundary conditions in equation (11) must be satisfied through the iterative solution of the system.

4 Numerical Experiments

In this section, we carry out several experiments to illustrate the accuracy and efficiency of the RBF interpolation in approximating credit default swap spreads. For the RBF interpolation, we employ the Thin Plate Spline (TPS) RBF as follows,

$$
\varphi (r_k) = r_k^4 \log (r_k),
$$

where $r_k = \| Z - z_k \|$ is the Euclidean norm. This particular RBF is chosen for two reasons. First, it does not require the calibration of additional parameters as some RBFs do (e.g. the Gaussian- and Multi-Quadratic (MQ)- RBF). Second, a previous study in option pricing by Koc et al. (2003) shows the outstanding performance of the TPS-RBF compared with the Cubic-, Gaussian- and MQ-RBF.

All experiments by the RBF interpolation employ 200 centers unless otherwise stated. We exploit the flexibility of this meshfree method by allocating centers over the space of the most concerned in the problem. Thus for pricing zero-coupon bonds we place the centers using the Halton points in the following way: for the one-factor model, we employ half of the centers in the domain $[0, 0.3]$ and the other half in the domain $[0, 2]$. Similarly, for the two-factor model, we use three groups with 25%, 25%, and 50% of the points over the spatial domains $[0, 0.05]^2$, $[0, 0.3]^2$ and $[0, 2]^2$, respectively. In order to approximate survival probabilities, we scatter the centers in both one- and two-factor models following a similar distribution to the one described above over the spaces $\ln([1E-05, 2])$ and $\ln([1E-05, 2])^2$, respectively. The discretization in time is carried out with 200 steps.

Regarding the FDM, we use a standard algorithm. For the one-factor model, the
regular grid is defined as \( N_x \times N_t \) with \( N_x = N_t = 200 \). In the two-factor model, we use the FDM with a grid of \( N_{x_1} \times N_{x_2} \times N_t \) with \( N_{x_1} = N_{x_2} = N_{x_3} = 200 \). The same features for the grids are applied to approximate the value of survival probabilities. We use the Crank-Nicolson averaging to discretize over time for both numerical approaches.

The numerical computations are performed on a DELL machine with Intel Core 2 Duo E8500 processor, CPU speed 3.17 GHz, internal memory 4.00 GB, hard drive disk with 160 GB and operating system Windows 7.

We define the approximation error as the distance \( \left( \hat{Y}^t - Y^t \right) \), where \( Y \) and \( \hat{Y} \) are the benchmark vector and the approximated values vector, respectively. Following Fasshauer et al. (2004) and Fasshauer (2007), we assess the performance of the numerical methods by two measures: the root mean square error (RMSE)

\[
RMSE = \sqrt{\frac{1}{L} \sum_{t=1}^{L} (\hat{Y}_t - Y_t)^2},
\]

and the absolute maximum error (ME)

\[
ME = \max \left( |\hat{Y}^t - Y^t| \right),
\]

where \( L \) is the total number of observations to be evaluated. From now on, for notational simplicity we rewrite the latent factors at time \( t = 0 \) in figures and tables in the following way: for the interest rate \( r^i_0 = x_i, i = 1, 2 \) and for the default intensity \( \lambda^j_0 = \exp(y_j), j = 1, 2 \).

### 4.1 Zero-Coupon Bond Price

We approximate the price \( P(x; 0, t) \) in equation (8) for one- and two-factor models for \( t = 1, 3, 5, 7 \) and 10 years. For the first factor \( r^1_0 \), we consider parameters \( \kappa_1 = 0.2, \theta_1 = 0.035 \) and \( \sigma_1 = 0.10 \). For the two-factor model, we further assume that \( \kappa_2 = 0.1, \theta_2 = 0.015, \sigma_2 = 0.05 \) and \( \rho_{1,2} = \rho_{2,1} = 0 \). The results of zero-coupon bond prices and their approximation statistics are reported as fractions of a nominal monetary unit of 1.
The closed-form solution is computed following Brigo and Mercurio (2006, equations (3.24), (3.25) and (4.38) on pages 66 and 176). The numerical solution is given by the approximation of the PDE (9) subject to the initial and boundary conditions (10) and (11), respectively.

Table 1 analyzes accuracy and computational efficiency of the numerical methods. We report zero-coupon bond prices with maturity $t = 5$ years and compute the RMSE and the ME. This experiment is run for several grid sizes for the FDM and different number of centers for the RBF interpolation. The CPU time in seconds is also shown. Panels A and B tabulate the results for one- and two-factor models, respectively. The statistics show that the RBF interpolation provides more accurate approximations than the FDM with a small number of centers. For instance, when the RBF interpolation is performed with just 50 centers for the one-factor model, the RMSE and the ME are smaller than those obtained by the FDM with a $300 \times 200$ grid. The accuracy of the RBF interpolation with 50 centers is narrowly defeated by the FDM with a $1000 \times 200$ grid, which takes 2.73 seconds to perform compared with only 0.02 second. For the two-factor model, the FDM generally becomes more expensive in terms of CPU time while the CPU time for the RBF interpolation does not increase very much compared with results in Panel A.

We also present a sensitivity analysis to investigate the stability of the numerical results against changes in model parameters in Figure 1. This is especially important for pricing derivatives in a market with turbulent conditions. The plots on the left examine the sensitivity of the approximated zero-coupon bond prices in terms of the latent factor $r_0^1$ and the CIR parameters. The plots on the right show the RMSE of the approximations by the RBF interpolation and the FDM. The analysis is conducted for the one-factor model with maturity $t = 5$ years. It is clear that there is a smooth transition in zero-coupon bond prices and the RMSE with respect to changes in parameter values.

In the Appendix, we provide further evidence of comparative approximation accuracy. Tables A1 and A2 report the analytical solution and approximation errors by the RBF interpolation and the FDM for the one- and two-factor models, respectively, for a selected sample of interest rate latent factors. We also extend our
accuracy analysis in terms of the RMSE and the ME to the domain $[0, 0.3]$ and $[0, 0.3]^2$ for interest rate for the one- and two-factor models, respectively, in Table A3.

4.2 Survival Probability

In the second set of experiments, we compute the probability $Q(y; \tau \geq t)$ defined in equation (14) for maturity $t = 1, 3, 5, 7$ and 10 years. We assume that the default intensity $\lambda_s$ follows an EV process according to equations (12) and (13). Under this assumption, there is no a closed-form solution. For the one-factor model, we set $\eta_1 = 0.1, \mu_1 = 0.02$ and $v_1 = 0.06$. For the two-factor model, we consider the parameters $\eta_2 = 0.2, \mu_2 = 0.005$ and $v_2 = 0.08$. Independence between factors is also assumed with $\gamma_{1,2} = \gamma_{2,1} = 0$. The approximated survival probabilities, approximation errors, the RMSE and the ME are all expressed as a fraction of unity.

The numerical solution for $Q(y; \tau \geq t)$ is given by approximating the PDE (15) subject to initial and boundary conditions (16) and (17), respectively. The approximation by the RBF interpolation is carried out by solving the system (22).

Table 2 provides a comparative analysis of accuracy and computational efficiency of the two numerical methods for the survival probability with $t = 5$ years. We summarize the RMSE, ME, and CPU time for several grid sizes for the FDM and different number of centers for the RBF interpolation. In this particular experiment, we use the approximation by the RBF interpolation with 500 centers as benchmark. Panels A and B report the numerical results for one- and two-factor models, respectively. This table shows that the RBF interpolation provides more accurate approximations than the FDM for different grid sizes. For example, for the one-factor model, the RMSE by the RBF interpolation with only 200 centers beat the FDM with a $1000 \times 200$ grid. The latter takes 2.73 seconds to compute while the former just 0.27 second. Similar conclusion can be drawn from the results in Panel B. The table shows that the CPU time required by the FDM for the two-factor model is are much longer. When the dimensional order increases, the advantage of adopting the RBF interpolation over the FDM increases as well.
Figure 2 evaluates the sensitivity of the approximations against changes in EV model parameters. The plots on the left examine the estimated survival probabilities while the plots on the right present the RMSE of the approximation by the FDM. The analysis is conducted for the two-factor model with maturity $t = 5$ years. We observe a smooth and gradual change in the survival probability and the RMSE in response to changes in parameter values. This analysis is important in evaluating the stability of the RBF interpolation against sudden changes in market conditions such as the recent credit crunch that started in 2007.

In the Appendix, we provide three more tables that summarize the approximation differences between these numerical methods for one-(Table A4) and two-factor (Table A5) models for a selected sample of latent variables. In Table A6, we extend our analysis to the domain $[0, 0.3]$ and $[0, 0.3]^2$ for the latent variables, respectively, for the one- and two-factor models to estimate the RMSE and the ME. These results all suggest that the differences between the FDM and the RBF interpolation increase with CDS contract maturity. This may result from the Crank-Nicolson discretization over time and generally it is more difficult to approximate values away from the initial condition than those close to the initial condition.

### 4.3 CDS Spread

In our last group of experiments, we compute the CDS spreads $R$ at time $0$ and maturity $t$ from equation (5). The values of zero-coupon bond prices and survival probabilities are based on the results obtained from the above experiments. We assume further that $L_{GD} = 60\%$ and the regular protection payments are quarterly $\alpha = 0.25$. The approximations by the RBF interpolation are taken as benchmark to compute the RMSE and the ME.

Table 3 summarizes the CDS spreads in basis points when we assume the two-factor models for the zero-coupon bond prices and default intensities, respectively. Panel A tabulates approximations by the RBF interpolation while Panel B summarizes pricing differences by the FDM. Table 4 reports the RMSE and the ME of CDS spread over the domain of $[0, 0.3]$ and $[0, 0.3]^2$, respectively, for the one- and two-factor models. The table shows that the differences in computed CDS spreads by
these two numerical methods are very small for the one-factor model and increase to some extent for the two-factor model.

In Figure A4 in the Appendix, we illustrate the CDS spreads by the RBF interpolation for the two-factor model over the surface of the problem. The figure shows that changes in default intensity have a greater impact on CDS spreads while the changes in the interest rate do not. This observation is important in the valuation of CDS spreads, in particular in turbulent market conditions with high levels of default probability such as the recent credit crunch.

So far we have considered cases that assume independence between factors. We now relax this restriction and introduce correlation between the latent factors $r_i^0$ and $\lambda_i^0$ with $i = 1, 2$. Figure A5 in the Appendix illustrates the differences in the CDS spreads under alternative correlation structures with respect to the benchmark when the factors are independent. We assume two dependence structures: positive with $\rho_{1,2} = \rho_{2,1} = 0.5$ and $\gamma_{1,2} = \gamma_{2,1} = 0.5$ (Panel A) and negative with $\rho_{1,2} = \rho_{2,1} = -0.5$ and $\gamma_{1,2} = \gamma_{2,1} = -0.5$ (Panel B). All approximations are implemented by the RBF interpolation. The plots show that differences are very small and suggest that in this framework the correlation between factors does not have a significant impact on the determination of CDS spreads.

5 Conclusion

In this study, we employ the meshfree radial basis function interpolation to approximate numerically the PDE describing zero-coupon bond prices and survival probabilities in a reduced-form model setting. In the numerical experiments, we assume that the interest rate follows a CIR process while the default intensity is modelled by the exponential-Vasicek process. We implement both one- and two-factor models. The results of the approximation are used to compute CDS spreads with and without correlation between the factors.

We find that the RBF interpolation outperforms the FDM in the terms of accuracy and computational efficiency, which are evaluated by means of the root mean square error and the absolute maximum error against benchmark and the CPU time.
needed. Hence, the RBF interpolation provides an overall superior performance than the FDM and our results are not sensitive to changes in parameter values. Therefore, we recommend the RBF approach as a valuable tool for researchers and practitioners in modelling credit derivatives where numerical approximation is needed, especially for multi-dimensional problems.

**References**


Table 1. Zero-Coupon Bond Price: Approximation Efficiency Analysis

(A) One-Factor CIR Model

<table>
<thead>
<tr>
<th>Centers $N$</th>
<th>RBF Interpolation</th>
<th>FDM $N_{x_1} \times N_t$</th>
<th>RMSE</th>
<th>ME</th>
<th>CPU Time</th>
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(B) Two-Factor CIR Model

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<th>FDM $N_{x_1} \times N_{x_2} \times N_t$</th>
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<th>CPU Time</th>
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Table 2. Survival Probability: Approximation Efficiency Analysis

(A) One-Factor EV Model

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(B) Two-Factor EV Model

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<th>RMSE</th>
<th>ME</th>
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**Table 3.** CDS Spreads: Two-Factor CIR $r$, Two-Factor EV $\lambda$ (in basis points)

### (A) RBF Interpolation: Approximated Solution

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<td>0.003</td>
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### (B) FDM: Approximation Error

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**Table 4.** CDS Spreads: Approximation Accuracy

### (A) One-Factor Models

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<th>ME</th>
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### (B) Two-Factor Models

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Figure 1. Zero Coupon Bond Price: Sensitivity Analysis

(A) $\kappa$: Speed of Reversion

(B) $\theta$: Long Term Mean Level

(C) $\sigma$: Instantaneous Volatility
Figure 2. Survival Probability: Sensitivity Analysis

(A) $\eta$: Speed of Reversion

(B) $\mu$: Long Term Mean Level

(C) $\upsilon$: Instantaneous Volatility