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Abstract

Constant Proportion Portfolio Insurance (CPPI) is a dynamic portfolio management strategy that is currently of popular interest in both industry and academic research. The CPPI methodology is designed to guarantee, to the buyer, a minimum payoff at maturity using a portfolio comprised only of one risky asset and one riskless asset. The goal is to allow an amount of participation in capital markets while removing downside risk. However, in the presence of realistic market assumptions risk to the seller exists in the form of *gap risk*. This gap risk accounts for the inability to meet the guarantee at maturity. There are many factors which contribute to the gap risk, including asset price behaviour and trading frequency. The effect of these factors are investigated in this paper within a discrete time framework.

The results show that when considering realistic levels of volatility the CPPI does not perform well in comparison to a riskless investment and a gapless (buy-and-hold) portfolio, respectively. CPPI returns are highly skewed and, in certain cases, fat-tailed. From the perspective of the buyer, the CCPI's higher expected return is based solely on the small chance of extremely large returns. In the majority of the cases, however, the CPPI yields a lower return than gapless and even riskless portfolios. Choosing an underlying with high volatility is even more hazardous since even the expected values are lower than that of the gapless alternative. These results are even more pronounced with the introduction of management fees where investors almost always can expect a lower return than from a corresponding buy-and-hold portfolio and typically fall substantially short of a riskfree investment. From the perspective of the issuer, a monthly rebalancing frequency is shown to be adequate to reduce the majority of the risk, while retaining a good payoff even when considering transaction costs.

Keywords: CPPI, Constant Proportion Portfolio Insurance, management fees, discrete trading, non-normal distributions, gap risk, return guarantees, capital guarantees.

JEL: C15, G11

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1 Introduction

Portfolio insurance strategies have been widely adopted as a risk management tool across the industry. They function to limit downside risk while at the same time offering an attractive upside potential. The interest and deployment of Constant Proportion Portfolio Insurance (CPPI) strategies, a particular implementation of portfolio insurance, has seen a resurgence recently with investor and client sentiment reflecting a more risk averse nature. CPPI based products offer security to the client through a predetermined guaranteed minimum payoff at maturity, while also giving the possibility of higher returns through limited exposure to a risky asset.

The CPPI method continuously rebalances capital between one riskless and one risky asset according to a clearly defined relationship. Devised by Perold (1986) and Black and Jones (1987) the strategy aims to guarantee at maturity at least the initial investment, plus any additional gains that the portfolio makes from its holding in the risky asset. The allocation of capital to riskless and risky assets is determined in the following way. A *floor* which grows at the risk-free rate to equal the guarantee at maturity is used to determine the *cushion*, which is equal to the difference between the current portfolio value and the current floor level. The *exposure* to the risky asset is calculated as the product of the cushion and the *multiplier*, with the remainder of the capital being allocated to the risk-free asset. The multiplier is a value (typically between 2 and 4) that can be considered as representative of the investor's risk aversion – the higher the multiplier the greater the investment in the risky asset. It is specified exogenously by the investor at the beginning of the investment and remains constant throughout the life of the product. Rebalancing of the portfolio occurs in reaction to movements in the risky asset with exposure being increased after a rise and decreased after a drop.

Originally portfolio insurance was achieved through the implementation of Option Based Portfolio Insurance (OBPI), which uses a listed put with the same maturity as the portfolio to cover an investment in a risky underlying, as introduced by Leland and Rubinstein (1976). However, the existence and availability of such options with the desired characteristics in the market is not guaranteed. This is particularly true when the risky assets are not equities or equity indices, but hedge funds and fund of hedge funds. The payoffs of such put options are typically synthesized by taking a position in the underlying and some riskless asset. This leads to the OBPI being implemented as a dynamic management strategy, which Bertrand and Prigent (2002) prove can be considered as a generalised form of the CPPI method where the multiplier is stochastic and variable over time. Nevertheless, the CPPI provides a more concise approach to portfolio insurance and explicitly caters for investor risk aversion through the multiplier.

Implementing CPPI as a viable dynamic portfolio strategy in practice presents many issues, relating to the risk of shortfall below the guarantee. This risk can be quantified as the probability that the floor will be violated during the portfolio's lifetime and the expected value of such a violation. In such a situation, it becomes impossible for the guaranteed payoff to be met, resulting in the issuer (or guarantor) having to make up the shortfall at maturity. This is known as the *gap risk* and its accurate quantification and subsequent pricing is key in the valuation and implementation of a CPPI product.

Under the continuous time CPPI model where the risky asset follows a continuous price process, there is no gap risk. However, in reality factors contributing to gap risk are liquidity, trading frequency and risky asset behaviour (i.e. various price process assumptions and their parameters), with the multiplier magnifying this risk. This paper considers discrete trading using a range of different rebalancing frequencies and multiplier values to ascertain their effect on the performance of the CPPI in conjunction with volatility, kurtosis, price jumps, leverage, management fees and transaction costs.

Bertrand and Prigent (2002) compare CPPI to OBPI over a number of different statistical criteria. On the moments of the two strategies, OBPI dominates in the mean-variance space. However, CPPI displays less downside risk and much stronger positive skewness. The dominance of OPBI over CPPI is shown to be more likely when it is an in-the-money call. At-the-money there is no clear dominance and out-of-the money the CPPI performs better. The delta and the gamma of the CPPI are always convex, and increasingly so for higher values of the multiplier, with the leveraged CPPI achieving values of delta above 1. The delta and gamma of the OBPI on the other hand follow that of a standard call. Since part of the attractiveness of a CPPI strategy is that it only requires the allocation of capital between a risky and riskless asset, comparing it to the synthetic put strategy is beneficial. Do (2002) does this for a number of consecutive 3 month periods (between 1992 and 2000) taken from an Australian index (AAOI: Australian All Ordinaries Index) and finds that the CPPI outperformed the synthetic put under daily rebalancing.

Perold and Sharpe (1988) explore the performance of a variety of dynamic investment strategies under different market conditions. They show that the CPPI favours strong bull markets, since this results in an increasing amount of capital being allocated to the growing risky asset. Alternatively, the CPPI protects in strong bear markets also, with capital being increasingly invested in the floor. It is markets that experience reversing trends that significantly impact the performance of the CPPI. Here, depending on the timing of rebalances, it is possible that the strategy sells on a drop which then rebounds or buys on a rise which subsequently presents a fall. Under such conditions they find a constant-mix strategy will have an advantage, while a buy-and-hold approach will be best when there is a large move in one direction. Clearly market behaviour can only really be known in hindsight, however Cesari and Cremonini (2003) find that the CPPI (and OBPI with a synthetic put) performs better than the other two strategies when market conditions are ignored. Annaert, Osselaer, and Verstraete (2006) find that that the CPPI offers a better downside protection and risk/return tradeoff than the buy-and-hold, but that this comes at the cost of lower excess return.

Balder, Brandl, and Mahayni (2005) investigate the effects of discrete time trading restrictions on the CPPI. They find that the discrete time performance of the CPPI is extremely sensitive to increases in the volatility of the underlying. Their result shows that a doubling of the volatility of the underlying causes the shortfall probability of the portfolio to increase by a factor of more than 50 with a multiplier of 12 and 12 rebalancing stages. Although they demonstrate that the performance of the discrete CPPI converges to that of the continuous time model as the rebalancing frequency increases, it is also made clear that the shortfall probability is not always a decreasing function of more frequent rebalances. In fact they show that as the number of rebalancing stages increases above 1, the shortfall probability increases until a critical number of rebalances is made.

With respect to discontinuities in the price process, Cont and Tankov (2007) investigate the effect of downward jumps in the risky asset price on the CPPI. They find that indeed downward jumps have a significant impact on the CPPI. However, for the jump parameters they estimate from two stocks (General Motors and Microsoft) the jump risk is not particularly high.

This paper is organised as follows: Section 2 discusses the models of the CPPI; Section 3 describes extensions to the standard model and outlines various processes used to model the dynamics of the underlying; Section 4 presents the results obtained; Section 5 concludes.

2 CPPI Models

2.1 Outline of CPPI Strategy

The CPPI is a guaranteed investment product which pays the buyer a predetermined minimum amount at maturity regardless of whether or not the value of the portfolio meets this minimum. In this sense the buyer has downside protection up to the guaranteed amount G, which is the minimum payoff they will receive at maturity T. Typically, the guarantee G is equal to the initial investment in the CPPI portfolio V_0, meaning that the buyer will at worst get back their original investment. At any time $t \in [0, T]$, the CPPI makes use of a reference to a floor value F_t which grows at the constant risk-free rate r, to equal the guarantee at maturity (see Equations (3) and (2)). Therefore the initial floor value F_0 is the guarantee discounted back at the risk-free rate from maturity. The value of the floor is used to calculate the cushion C_t, which is the difference between the value of the portfolio at any time V_t and the floor F_t as shown in Equation (1). Recall that within the CPPI portfolio, investment of the capital is split between two assets: one risk-free asset B_t which follows the same dynamics as the floor (see Equation (6)) and one risky asset S_t (the properties of which are discussed later). The CPPI first determines the amount of risky asset to invest in and then allocates the remaining capital to the riskless asset. The exposure E_t to the risky asset is the product of the cushion and a constant multiplier value m, as described in Equation (4), leaving B_t to be invested in the riskfree asset (see Equation (5)). The basic relationship can be summarised as follows:

$$V_t = F_t + C_t \tag{1}$$

$$F_t = Ge^{-r(T-t)} \tag{2}$$

$$dF_t = rF_t dt \tag{3}$$

$$E_t = mC_t \tag{4}$$

$$B_t = V_t - E_t \tag{5}$$

$$dB_t = rB_t dt \tag{6}$$

To illustrate the workings, consider an example, where $m = 3$ for a unit investment in a CPPI with a 5 year maturity, constant risk-free rate of 5% and guarantee equal to the initial capital. At time $t = 0$, the initial value is (by assumption) $V_0 = 100\%$ which is also the guaranteed amount G the investor will receive at maturity. The initial floor is $F_0 = 100\% \cdot e^{-0.05 \cdot 5} = 77.88\%$, leaving an initial cushion of $C_0 = 100\% - 77.88\% = 22.12\%$. With a multiplier of $m = 3$, the initial exposure is $E_0 = 3 \cdot 22.12\% = 66.36\%$. Therefore at the start of the investment, $B_0 = 100\% - 66.36\% = 33.64\%$ of the initial capital is allocated to the risk-free asset and the remaining $E_0 = 66.36\%$ is invested into the risky asset. In the course of time, price changes of the risky underlying will affect V_t, and allocated amounts need to be rebalanced so that the basic relationship holds again.

It is clear that increasing m shifts more capital from the risk-free asset to the risky asset subject to restrictions on the amount of leverage allowed. Leverage constraints are often imposed on the CPPI to prevent additional capital being borrowed outside of the initial investment. When leverage in not permitted then the portfolio is said to be unlevered or self-funding. In this case, revisiting the previous example and setting $m = 5$, the exposure becomes equal to 110.6% which exceeds the initial capital and so this would be capped at 100%. The value of this multiplier is stated from the outset of the CPPI strategy and remains constant throughout the life of the investment. The higher the multiplier value the greater the exposure to the risky asset and the greater the potential for larger gains and losses to be realised.

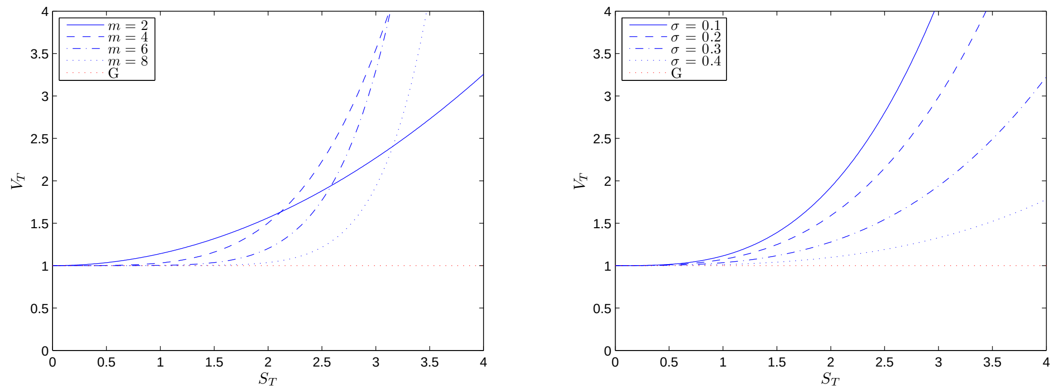
A description of how the CPPI strategy alters the portfolio composition over time is now presented. If the return on the risky underlying is greater than the risk-free rate r, then the cushion will increase in value and more capital will be allocated to the risky asset. This increase in investment in the risky asset is funded by a reduction in the amount of risk-free asset held. Similarly the cushion shrinks when the risky asset experiences a loss or grows by less than the risk-free rate in any period of time. This causes some of the risky asset to be sold with the proceeds being invested into the riskless asset. Since the risk-free rate is constant, the progression of the CPPI is entirely dependent on the stochastic nature of the risky asset. With respect to different multiplier values the behaviour of the CPPI is as follows. In the case that $m = 1$, the CPPI can be considered *gapless* in that there no rebalancing is required; hence, there is a 100% certainty that the portfolio will not gap. Essentially, with $m = 1$ the portfolio becomes a buy-and-hold investment since the exposure is equal to the cushion and risk-free investment is equal to the floor which grows to meet the guarantee at maturity. The performance of the gapless portfolio is determined entirely by the terminal asset price S_T and is not path dependent. However, it is for $m > 1$ that the CPPI is an interesting strategy and this is also when the risk of gapping is introduced.

2.2 CPPI under Continuous Time and Lognormal Price Process

The continuous time model for CPPI was derived by Black and Jones (1987) under the assumption that the risky asset follows a geometric Brownian motion (GBM). The model assumes that no market frictions are present and that there is no limit on the amount of leverage that may be applied (i.e. extra capital may be borrowed at the risk-free rate). The value of the CPPI portfolio V_t can be separated into a stochastic and non-stochastic component as already shown in Equation (1), comprising of the cushion and floor respectively. Equation (7) gives the value of the CPPI portfolio at time t as a function of the floor and cushion process. It is clear that the current stock price S_t is the only stochastic factor driving the CPPI once the other parameters have been determined initially,

$$V_t = F_t + (V_0 - F_0)\left(\frac{S_t}{S_0}\right)^m \exp\left\{\left(r - m\left(r - \frac{\sigma^2}{2}\right) - \frac{1}{2}m^2\sigma^2\right)t\right\} \tag{7}$$

The volatility of the risky asset per year is denoted by σ. Note, since the cushion process cannot become negative, the value of the portfolio never falls below the floor and



(a) $\sigma = 0.2$, various m (b) $m = 3$, various σ

Figure 1: Continuous time lognormal CPPI: $\mu = 0.1$, $r = 0.05$, $T = 5$

the portfolio never gaps. Additionally, the process is path independent. Appendix A provides a complete derivation of Equation (7). The terminal payoff of the portfolio becomes steeper with increasing m as seen in Figure 1(a). Although, for example, the $m = 8$ portfolio payoff does not dominate the $m = 2$ portfolio for all values of S_T, for larger values of S_T (3 to 3.5) the growth is substantially higher. Figure 1(b) shows the payoffs increasing with S_T for decreasing values of σ. It should be noted that although the graph demonstrates that a lower volatility will result in a higher terminal portfolio payoff for the same value of S_T, the expected value of V_T is the same for any value of σ (see Equation (8)) i.e. σ does not effect the expected payoff.

The expected value and variance of the CPPI under a continuous time lognormal price process are (see Appendix A for derivation)

$$E[V_t] = F_t + (V_0 - F_0)\exp\{(r + m(\mu - r))t\} \tag{8}$$

$$Var[V_t] = (V_0 - F_0)^2 \exp\{2(r + m(\mu - r))t\}\left(\exp\{m^2\sigma^2 t\} - 1\right) \tag{9}$$

which gives the expected payoff of the CPPI as an increasing function of m and independent of volatility.

2.3 Unconstrained Generic Discrete Time CPPI

Under discrete time it is assumed that the price of the risky asset is observed at discrete intervals and that rebalancing of the portfolio only occurs at these times. At time t the change in the CPPI portfolio value satisfies the following (see Prigent (2005)):

$$dV_t = (V_t - E_t)\left(\frac{dB_t}{B_t}\right) + E_t\left(\frac{dS_t}{S_t}\right).$$

With discrete changes this can written as

$$\Delta V_t = (V_t - E_t)\left(\frac{\Delta B_t}{B_t}\right) + E_t\left(\frac{\Delta S_t}{S_t}\right).$$

Borrowing notation from Balder et al. (2005) the discrete time scale for an investment with a horizon of T is defined across the interval $[0,T]$ with n equidistant points as belonging to τ such that

$$\tau = \{0 = t_0 < t_1 \ldots < t_{n-1} < t_n = T\}$$

where

$$\Delta t = t_k - t_{k-1} = \frac{T}{n} \qquad k = 1,\ldots,n.$$

Since the CPPI portfolio is rebalanced every period, $n-1$ represents the number of rebalances made during the life of the investment after the initial setup. A generic CPPI, with no assumption on the process driving the risky asset can be described in discrete time as

$$V_{t_k} = (V_{t_{k-1}} - E_{t_{k-1}})e^{r\frac{T}{n}} + E_{t_{k-1}}\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right) \tag{10}$$

$$F_{t_k} = F_0 e^{r\frac{T}{n}k} \tag{11}$$

with

$$e^{r\frac{T}{n}} = \frac{B_{t_k}}{B_{t_{k-1}}}$$

and as before

$$F_0 = Ge^{-rT}.$$

The growth of the portfolio can be seen to be dependent on the growth of the exposure and risk-free investment, given by the risky asset's return and the risk-free rate ($e^{r\frac{T}{n}}$) respectively. Equation (10) demonstrates the path dependence of the model, whereby the current value of the portfolio is dependent on the previous period's level of exposure and the risky asset's return. Substituting the discretized version of Equations (1) and (4) into (10) gives

$$V_{t_k} = \begin{cases} F_{t_k} + (V_{t_{k-1}} - F_{t_{k-1}})\left(m\frac{S_{t_k}}{S_{t_{k-1}}} - (m-1)e^{r\frac{T}{n}}\right) & \text{if } V_{t_{k-1}} \geqslant F_{t_{k-1}} \\ V_{t_{k-1}} e^{r\frac{T}{n}} & \text{if } V_{t_{k-1}} < F_{t_{k-1}} \end{cases} \tag{12}$$

showing that, regardless of the underlying price process governing S, when the multiplier value $m = 1$ it is not possible for the floor to be violated under any circumstance. For values of $m > 1$ however, there is a risk that the floor will be violated, dependent on the movement of S. Given that the floor has not previously been breached, V_{t_k} will drop below the floor if[1]

$$\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} > \left(1 - \left(1 - \frac{1}{m}\right)e^{r\frac{T}{n}}\right). \tag{13}$$

If the floor is breached then the value of the portfolio grows at the risk-free rate.

[1] Note that this is only true when there is no constraint on the amount of leverage that may be used; see Section 2.5.

2.4 Alternative Price Processes

As discussed previously the progression of the CPPI depends on the stochastic behaviour of the risky asset. Therefore by considering different processes driving the risky asset's price, the effects of these processes and their attributes can be studied. The assumption of the classic CPPI analytical model is that the risky asset follows a lognormal price process as follows:

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\} \tag{14}$$

where μ is the expected rate of return, σ is the volatility of the price and W_t is a Wiener process. As discussed previously, under a continuous time assumption an asset following a lognormal price process does not generate any gap risk for the CPPI. However, in the presence of discrete rebalancing stages this does not hold true.

Well documented in financial literature is the failure of the geometric Brownian motion model to capture the fat tails that are evident in returns distributions. The Normal distribution, used to model returns, does not exhibit a high enough kurtosis to realistically represent stock returns. The Student t distribution, however, does allow a varying level of kurtosis to be captured through the manipulation of its *degrees of freedom* parameter. By replacing the Normal distribution with Student t within a GBM, a geometric random walk is achieved producing *log-t* distributed prices. The price process is therefore defined as

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\left(\frac{\nu-2}{\nu}\right)D_t\right\} \tag{15}$$

where ν represents the number of degrees of freedom and $(\nu-2)/\nu$ scales the effect that the Student t process in D_t has on the standard deviation.

The existence of random jumps in the movement of prices, often attributed to arrival of significant new information, produces heavy tails when included in a model. Discontinuities in the asset price are of concern to investors and particularly so for CPPI type products since jumps in price movements could lead to portfolios suddenly falling below the floor. The jump diffusion model of Merton (1973) can be viewed as an asset following a GBM with the addition of a jump component J:

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW_t + dJ_t,$$

where S_{t^-} is the price at t immediately before a jump is realised. The solution to this equation can be stated as

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \sum_{i=1}^{P_t} j_i\right\} \tag{16}$$

where P_t represents a Poisson process that generates integers at a rate of λ, which is the expected number of jumps per year. It follows that up to time t, $E[P_t] = \lambda t$. j_i is an i.i.d random variable that models the size of the jumps. When j_i is normally distributed ($j_i \sim N(a,b)$), S_t is distributed as follows (compare (Glasserman, 2003, p.136)):

$$\ln S_t \sim N\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + a\lambda t, \sigma_j\sqrt{t}\right).$$

where the volatility of the jump diffusion process is

$$\sigma_j = \sqrt{\sigma^2 + b^2\lambda}. \tag{17}$$

If $j_i \sim N(0, b)$ then the model produces symmetrical jumps about 0 and the jumps do not effect the mean of the price.

2.5 Constraints on Shorting and Leverage

Constraints to the CPPI model are now considered to bring it in line with what is realistically possible in an operational setting. From the basic relationship of the CPPI in Equation (1), it is evident that in the event of the portfolio value dropping below the floor the cushion becomes negative. This is not a desirable effect since it leads to a negative exposure, implying shorting of the risky asset which is often prohibited in practice. To prevent this the cushion is restated as

$$C_{t_k} = \max[(V_{t_k} - F_{t_k}), 0]. \tag{18}$$

Regulations governing the operation of funds impose limits on the amount of leverage that may be used. If the product is deemed to be self-funding then no additional capital may be borrowed externally and the maximum exposure is limited to the value of the fund. A constrained exposure whereby the maximum possible exposure is a percentage h of the current portfolio value is

$$E_{t_k} = \min[mC_{t_k}, hV_{t_k}]. \tag{19}$$

For example, for a self-funding product the leverage is zero and $h = 1$, allowing a maximum of 100% of the portfolio capital to invested in the risky asset. A value of $h = 2$ implies that a maximum exposure of 200% of the portfolio value is allowed and the portfolio is said to have a maximum leverage of $h - 1 = 100\%$.

2.6 Transaction Costs and Management Fees

Transaction costs are considered as a proportional cost levied on the amount of the risky asset traded whenever the portfolio is rebalanced:

$$TC_{t_k} = |E_{t_k^+} - E_{t_k}| \cdot \eta \tag{20}$$

$$0 < \eta < 1 \tag{21}$$

where E_{t_k} is the amount invested in the risky asset after observing the risky asset's growth for the period t_{k-1}^+ to t_k and $E_{t_k^+}$ is the new amount allocated for the next period according to Equation (4). The difference gives the quantity of risky asset traded. In Equation (20) η is the percentage transaction cost applied to the trade.

In an operational setting institutions charge management fees which are quoted as an annualized percentage of the CPPI portfolio value. This fee is calculated and typically deducted from the portfolio value on a daily basis. For computational simplicity it is assumed that the fee is taken at each rebalancing point as follows:

$$Fee_{t_k} = \begin{cases} \frac{\phi}{p} \cdot V_{t_k} & \text{if } V_{t_k} \geqslant F_{t_k}/(1 - \frac{\phi}{p}) \\ 0 & \text{if } V_{t_k} < F_{t_k}/(1 - \frac{\phi}{p}) \end{cases} \tag{22}$$

where $\phi \geq 0$ is the annualized percentage fee and p is the number of periods in a year. The fee per period is simply the annualized fee divided by the number of periods p in a year. If a deduction of the fee would result in the floor being violated at t_k then no fee is charged at this time. It is assumed that it would not be beneficial to the decision maker to take a fee causing a shortfall at time t_k only to repay this amount at maturity to provide the guarantee.

3 Computational Study

3.1 Experiment Design

To test the effects of different distributions and market frictions, simulations of the CPPI are undertaken with the following assumptions. The risky asset is liquid and can be traded on the rebalancing date at the current price provided by the simulation. For underlyings such as prominent stock indices (e.g. FTSE 100), liquidity can be assumed to be daily since these are actively traded products. For less liquid underlyings (e.g. hedge funds), reduced frequency in portfolio rebalances can be argued to be reflective of the fact that the underlying can be only rebalanced at that frequency. Both the risky and riskless asset can be traded in infinitely divisible amounts. This is not unrealistic considering that most CPPI funds have capital in excess of \$100 million and the impact of having to buy or sell individual units of risky asset or bonds becomes insignificant. The riskless asset can be considered analogous to a bank account with negligible transaction costs, so they are ignored here. However, they are considered for the risky asset. Shorting of the risky asset is not permitted. A unit investment is considered, giving a terminal log-payoff $\ln(V_T)$ equal to the accumlated log-return.

Unless otherwise stated the model assumes a GBM for the underling risky asset, 5 year maturity ($T = 5$), monthly rebalancing (i.e. 12 periods per year $p = 12$ or 60 rebalances during investment's life of 5 years: $n = 60$), no leverage (i.e. $h = 1$ in Equation (19)), no transaction costs, no management fees and a volatility of 20% ($\sigma = 0.2$). The risk-free rate and the expected rate of return are always fixed at 5% and and 10% respectively. These parameter values can be considered typical for a CPPI with a reasonably liquid underlying. Note that the discrete CPPI rule in Equation (12) has not been updated to reflect the constraints outlined in the two previous subsection, but the constraints have been implemented.

Results published in this paper are the outcome of 10^6 simulations for each respective price process and parameter set. This number of simulations has been deemed a suitable number when considering the tradeoff between convergence of results and computational execution time.

3.2 Statistical Tests

To capture the statistical properties of the CPPI from the perspective of the issuer, the following measures have been used:

- Moments of the log terminal portfolio value V_T of all portfolios and the log terminal portfolio value V_T^L of those portfolios that experienced a loss. V_T^L is defined as:

$$V_T^L = [V_T | V_T < G]. \tag{23}$$

- Distribution of the losses L_T, where a loss is taken as the amount the portfolio value is below the guarantee at maturity, given that it is below the guarantee:

$$L_T = [G - V_T | V_T < G]. \tag{24}$$

- Percentage of times a loss occurs, which when measured over a large number of simulations can also be interpreted as the probability that the CPPI will experience a loss:

$$Pr[L_T] = \frac{1}{I} \sum_{i=1}^{I} \mathbb{B}_{(V_{T,i} < G)} \tag{25}$$

where I is the number of simulations performed and $\mathbb{B}$ is a binary indicator function returning 1 if the subscript term is satisfied and 0 otherwise. Note that $\Pr[L_T]$ can be interpreted from the buyer's perspective as the probability that they will receive only the guarantee at maturity.

- Expected value (expected shortfall) of the loss below the guarantee;

$$E[L_T] = E[G - V_T | V_T < G]. \tag{26}$$

The expected values and medians of V_T^c / V_T^{rf} and $V_T^c / V_T^{m=1}$ are taken to capture the value of the CPPI at maturity in relation to the risky underlying and riskfree/gapless portfolios from the perspective of the buyer (client). The terminal value of a 100% risk-free investment is given by Equation (27) and Equation (28) gives the value of the gapless portfolio ($V_T^{m=1}$) which is the same as a CPPI when $m = 1$. V_T^c is the value of the portfolio from the buyer's perspective, meaning its value at maturity is at least the guaranteed amount and is defined in Equation (29).

$$V_T^{rf} = V_0 e^{rT} \tag{27}$$

$$V_T^{m=1} = G + (V_0 - F_0)\frac{S_T}{S_0} \tag{28}$$

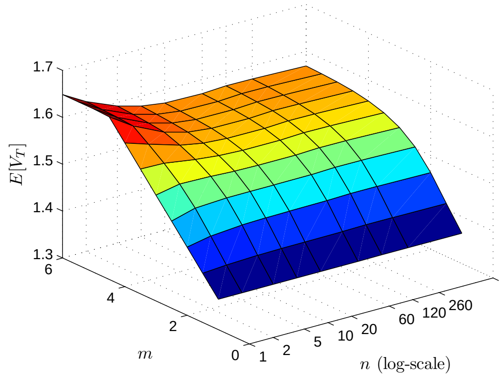
$$V_T^c = \max(V_T, G). \tag{29}$$

3.3 Impact of Trading Frequencies

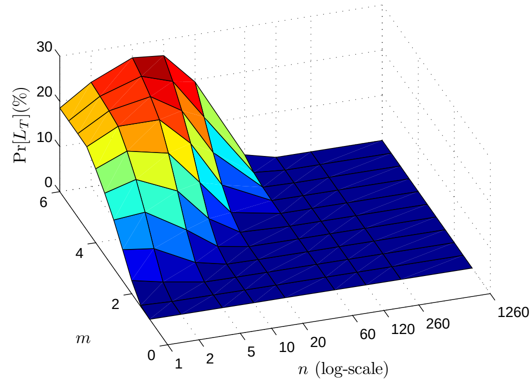
The frequency at which the CPPI portfolio is rebalanced is important since it is not desirable to trade more often than necessary, especially when costs are present. Conversely, not rebalancing the portfolio often enough increases the risk. To determine a reasonable rebalancing frequency, the CPPI has been simulated with the following number of rebalancing points n in the 5 year investment: $n \in \{1, 2, 5, 10, 20, 60, 120, 260, 1260\}$, where $n = 1$ corresponds to a buy-and-hold investment, $n = 5$ is yearly rebalancing, $n = 20$ is quarterly, $n = 60$ is monthly, $n = 260$ is weekly and $n = 1260$ is daily.

Figure 2 shows that the $E[V_T]$ for lower m ($m \leqslant 2.5$) quickly converge to a maximum as the trading frequency increases, with $n > 20$ causing little additional change in the payoff. For $m \geqslant 4.5$ the best payoff is achieved for very low trading frequencies. This is because for $m \geqslant 4.5$, the exposure is at 100% initially causing the expected portfolio value to grow in line with the expected value of the risky asset up until the first rebalancing point. Therefore with low rebalancing frequencies, the relatively long time until the first rebalancing point (for $n > 1$) means the portfolio benefits from the positive expected growth, which can add substantial value to the portfolio in this time. When $n = 1$ the growth of the portfolio mimics that of a 100% buy-and-hold investment in the risky asset, meaning that all portfolios with $m \geqslant 4.5$ produce the exact same result. Low trading frequencies also expose the CPPI to increased risk as indicated by the $\Pr[L_T]$ and $E[L_T]$ surfaces. The $\Pr[L_T]$ surface in Figure 2(b) has a hump around the yearly ($n = 5$) rebalancing mark, the shape of which is not reflected in the $E[L_T]$ surface of Figure 2(c). This shows that yearly rebalancing results in a higher probability of loss than the buy-and-hold ($n = 1$), but the expected value of this loss is significantly lower. The reason for the humped effect is as follows. For a buy-and-hold investment of $n = 1$ the value of the portfolio is only observed at maturity where it can be ascertained whether or not it has met the guarantee. Since the portfolio is not rebalanced during the investment a substantial number of losses are realised. However, moving to yearly rebalancing, more losses are realised because the portfolio is rebalanced and therefore checked more often. As an example, consider if the portfolio has violated the floor just before the end of year 2 then all the capital will be invested in the risk-free asset ensuring the CPPI will never meet the guarantee. Had the same situation occurred with the buy-and-hold investment, there existed a chance for the portfolio to recover in the remaining 3 years until maturity. As n increases beyond yearly trading, $\Pr[L_T]$ is decreasing because now the portfolio is monitored sufficiently frequently to prevent large movements in the risky asset from occurring before the portfolio can be rebalanced. From Figures (2(b)) and (2(c)) it can be demonstrated that monthly rebalancing ($n = 60$) is frequent enough to eliminate the majority of the risk.

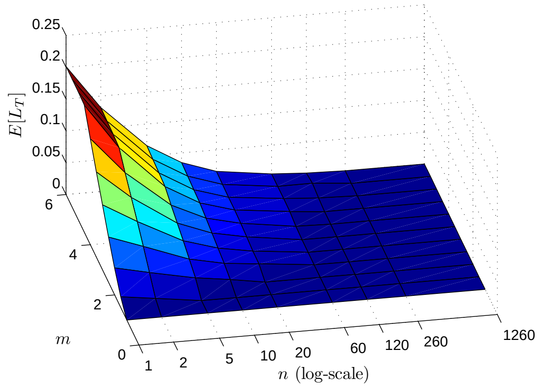
Figure 3 compares the terminal portfolio value against the terminal risky asset value for daily and quarterly rebalancing. The continuous time portfolio is also plotted for reference. Under daily rebalancing the discrete time CPPI payoffs lie close to the analytical solution. With quarterly rebalancing however, there is a large deviation either side of the continuous time plot. For quarterly rebalancing the root mean square deviation (RMSD) of the discrete terminal portfolios from the continuous time analytical solution is calculated as 0.2867, while for daily rebalancing it is 0.0354. Thus quarterly rebalancing produces a very high dispersion around the analytical solution compared to daily trading. However, note that some of the simulations under quarterly rebalancing produce very high payoffs that are not matched by daily rebalancing. This occurs because when rebalancing less frequently the risky asset price is observed less frequently, and it is more likely that a monotonically increasing S will be observed at quarterly intervals ($n = 20$) than daily intervals ($n = 1260$). Couple this with no constraint on leverage, the opportunity for very high payoffs to be achieved is possible, although very rare.



(a) Expected terminal portfolio value V_T

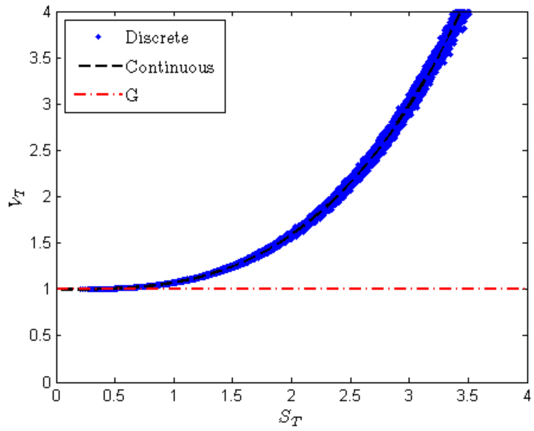


(b) Probability of losses L_T

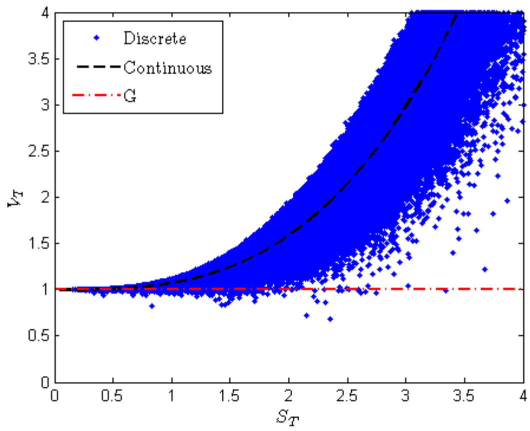


(c) Expected value of losses L_T

Figure 2: Effect of varying trading frequencies and multiplier values



(a) Daily rebalancing



(b) Quarterly rebalancing

Figure 3: Comparison of terminal payoffs V_T against S_T, unconstrained leverage

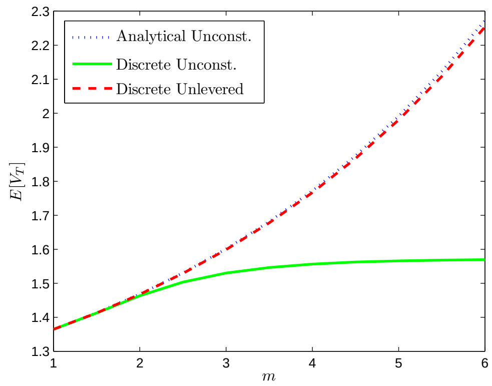


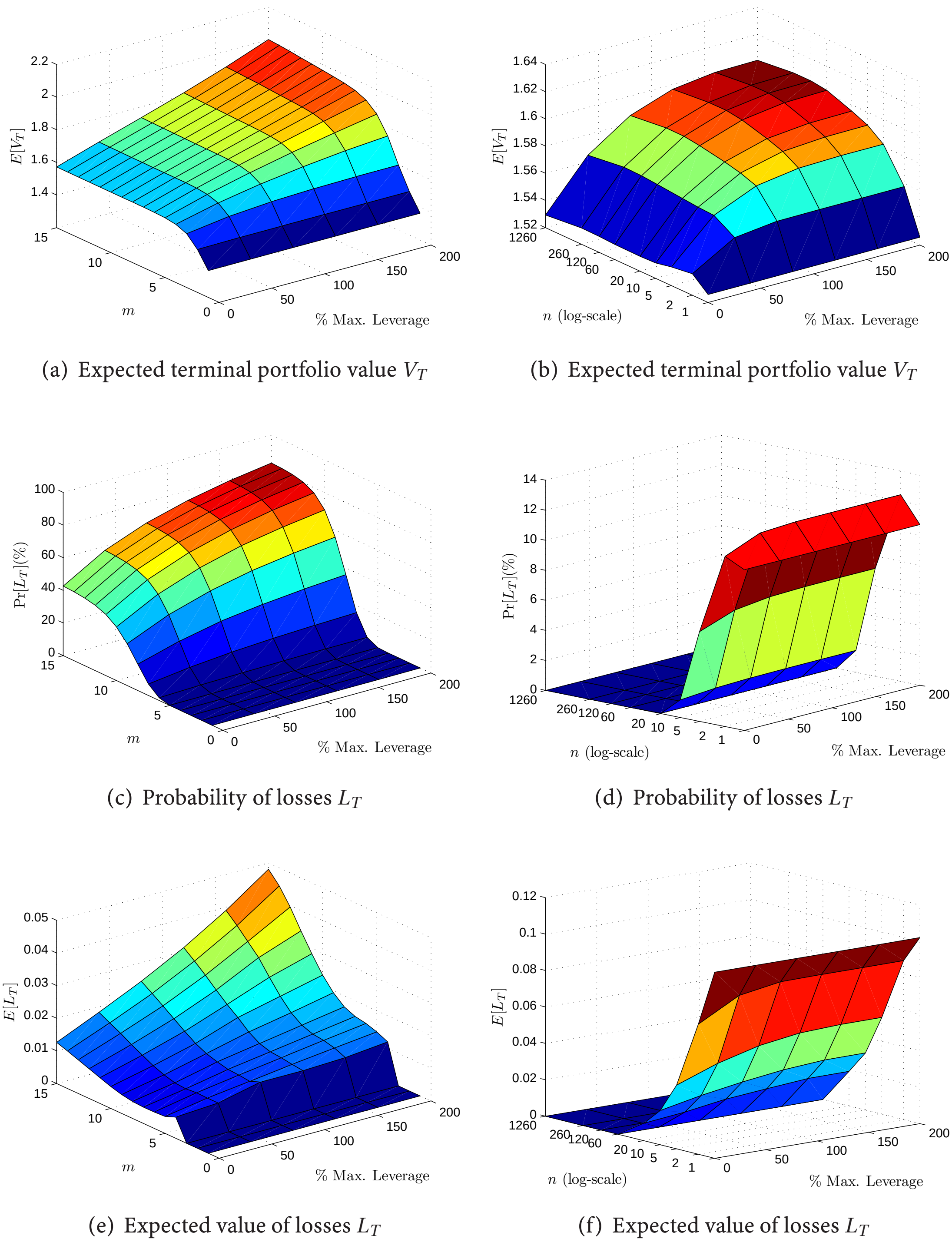
Figure 4: Comparison of leverage effects on $E[V_T]$ in continuous and discrete time

3.4 Leverage Effects

With no limit on the amount of leverage and assuming that capital may be borrowed at the same rate as the risk-free investment, Figure 4 shows that the unconstrained discrete time CPPI closely follows the analytical solution for various values of m. Both these models show the $E[V_T]$ is growing exponentially with m. In contrast and as would be expected, the CPPI with no leverage approaches a limit as a result of being constrained by its initial capital.

Figures 5(a), 5(c) and 5(e) demonstrate the outcome of leverage and varying m on the CPPI. Values of $m = 1 \dots 15$ have been used to ensure that the extra potential leverage is utilised by the CPPI. The $E[V_T]$ increases with leverage and multiplier value as shown in Figure 5(a). For all values of the leverage, the surface increases steeper at first and then less so as m increases. The higher the leverage, the higher the m value at which this change occurs. This effect is the result of the higher ms exploiting the higher leverages. However, the limit on the maximum leverage means that the exposure to the risky asset is also limited and instances where the risky asset's return is very high are not fully benefited from. This is why $E[V_T]$ increases linearly for all leverages once a certain value of m is reached. Compare this to the unconstrained CPPI in Figure 4, where the expected payoff is always convex. Figure 5(c) shows that the probability of loss quickly increases with both m and leverage. For a maximum leverage of 200% and $m = 12$ there is approximately an 80% chance of shortfall occurring, which is particularly high. However, looking at Figure 5(e) it can be seen that the expected value of this loss will only be around 3.5% of the initial portfolio value and these frequent losses do not appear to impact the $E[V_T]$ which is always increasing with m and leverage. Additionally, it is reasonable to consider that the impact of large outliers in V_T that arise from the use of leverage, skew the distribution which in turn prevents the mean being reduced by the losses.

The graphs in Figures 5(b), 5(d) and 5(f) show that leverage against rebalancing frequency produces a concave surface for $E[V_T]$. When there is no leverage, increasing trading frequency has little impact on $E[V_T]$ which is relatively flat apart from a peak at $n = 2$. This effect is present in Figure 2(a), but not easily seen given the scale. The effect is seen because when $n = 1$, the portfolio is a buy-and-hold investment and



(a) Expected terminal portfolio value V_T

(b) Expected terminal portfolio value V_T

(c) Probability of losses L_T

(d) Probability of losses L_T

(e) Expected value of losses L_T

(f) Expected value of losses L_T

Figure 5: Effect of leverage in conjunction with varying multiplier values m (left; $n = 60$) and rebalancing frequencies n (right; $m = 3$)

with $m = 3$ approximately 66% of the capital is allocated to the risky asset. This is not enough to benefit greatly from the growth in the risky asset. When $n = 2$, the portfolio is rebalanced half-way through the life of the investment, at which point exposure to the risky asset is increased for those portfolios where the risky asset performed well. Here, strongly performing portfolios cause the mean payoff to be increased. The same behaviour is not seen for $n > 2$ because as the number of rebalancing points increases the probability of successive points seeing strong growth drops. Therefore exposure to the risky asset is not consistently increased, leading to many more lower payoffs which reduce the mean. When no leverage is allowed the growth of strongly performing portfolios is limited and their values are not high enough to lift the mean. However for maximum leverages of 50% and above, the increase in the number of rebalances has a clearly positive impact on the expected payoff. With the extra capital available, those portfolios with a 100% investment in a strongly performing risky asset will have very high payoffs which cause an increase in the mean.

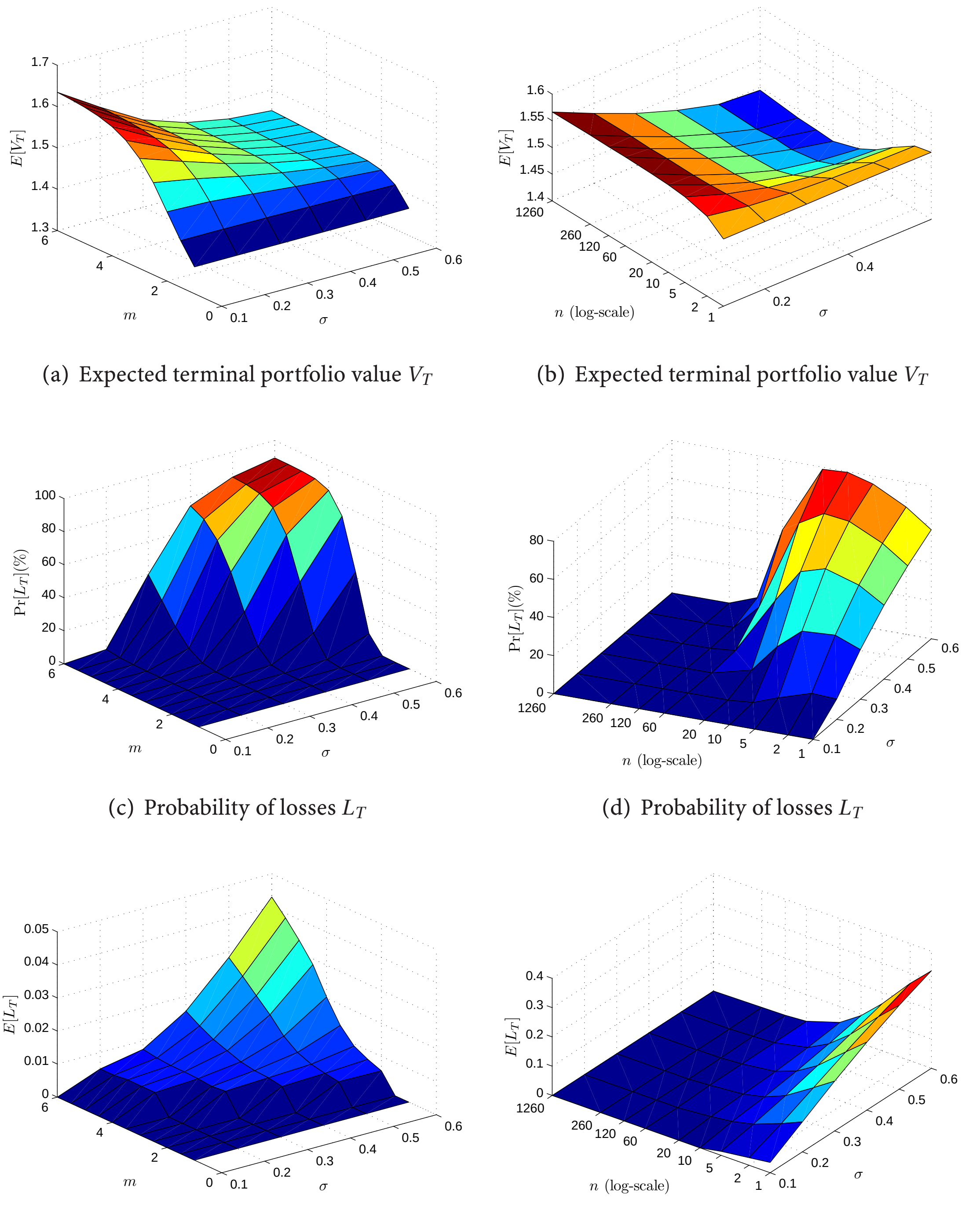
Figures 5(c) and 5(e) show that the leverage has very little effect on the risk of the CPPI when the multiplier is fixed at $m = 3$. Even though leverages up to 200% are investigated this size multiplier is unlikely to exploit this leverage in the majority of simulations and it is only the outliers which cause an increase in the mean.

3.5 Volatility Effects

This section explores the effect of differing volatilities on the CPPI. In the continuous time analytical model, it was noted in Equation (8) that the expected value of the CPPI was independent of volatility. However, as Figure 6(a) clearly demonstrates, increases in volatility have a detrimental effect on expected value of V_T, which is further amplified by large values of m. For higher values of volatility, increases in m produce little additional payoff. The number of losses can also be seen to be increasing with volatility to over 90% for $m = 6$ and $\sigma = 0.6$ (Figure 6(c)), but the expected value in comparison is still relatively small at about 5% of V_0 (see Figure 6(e)).

Figures 6(b), 6(d) and 6(f) show the changes observed when both volatility and trading frequency are varied. For the expected value of V_T it can be seen that as the volatility of the underlying increases, $E[V_T]$ decreases. However, although the payoff improves with increased trading frequency for low volatilities, it is clear that for higher volatilities an increase in trading is detrimental to the performance of the CPPI. This can be explained by the fact that with a low number of rebalances the investment behaves more like a buy-and-hold strategy and the CPPI is less effected by large up and down movements. Looking at the graphs for the probability and expected value of losses reveals that more frequent trading does greatly reduce both the number and size of losses, indicating that more frequent trading would still be preferred. Indeed it should be noted that the $E[L_T]$ for low rebalancing frequencies is particularly high.

Monthly trading ($n = 60$) can be said to be a reasonable trading frequency to largely eliminate risk for volatilities up to 30%, but weekly rabalancing ($n = 260$) is better when dealing with volatilities up to 60%. Examining both Figures 6(e) and 6(f) shows that rebalancing frequency has a greater effect on risk when coupled with volatility than multiplier value is (the $E[L_T]$ of Figure 6(f) has a range approximately 4 times greater than that of Figure 6(e)). This suggests that it is more effective to trade more frequently



(a) Expected terminal portfolio value V_T

(b) Expected terminal portfolio value V_T

(c) Probability of losses L_T

(d) Probability of losses L_T

(e) Expected value of losses L_T

(f) Expected value of losses L_T

Figure 6: Effect of volatility in conjunction with varying multiplier values m (left; $n = 60$) and rebalancing frequencies n (right; $m = 3$)

than to lower m to reduce risk during volatile periods, but this comes at the expense of a lower $E[V_T]$.

3.6 Kurtosis Effects

The significance of kurtosis on the CPPI is investigated through the simulation of the risky asset using a Student t geometric random walk with varying degrees of freedom (ν). The effects of the following degrees of freedom are investigated: $\nu = 15$, $\nu = 10$ and $\nu = 7$, which approximately correspond to a kurtosis of 3.5, 4 and 5 respectively.

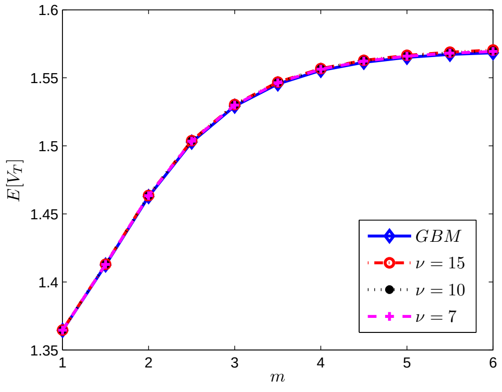
Figures 7(a), 7(c) and 7(e) show the effect of the different degrees of freedom in comparison to the GBM process for varying multiplier values. The expected payoffs are indistinguishable from one another indicating that kurtosis has no effect on this aspect of the CPPI. However, the results for the $\Pr[L_T]$ and $E[L_T]$ show that as kurtosis increases, the probability and size of losses also increases. Unusually the $E[L_T]$ decreases as m increases for the Student t processes. This effect is the result of the higher values of m increasing exposure to the risky asset, and coupled with the symmetry of the Student t process, larger losses are offset by large profits causing the size of losses to be reduced. For values of $2 \leqslant m \leqslant 4$, $E[L_T]$ is at its highest because the exposure to the risky asset is high enough for losses in the risky asset to cause a shortfall, but not high enough to fully benefit from any larger profits. The symmetry of the Student t process also explains why the $E[V_T]$ plots are very close for varying levels of kurtosis.

Figures 7(b), 7(d) and 7(f) illustrate how rebalancing frequency effects the CPPI with increasing levels of kurtosis. The payoffs achieved by Student t process CPPI follow closely that of the GBM, with a trend showing higher levels of kurtosis resulting in marginally higher payoffs. Figure 7(d) also shows that higher kurtosis produces higher probability of loss, which reduces as rebalancing frequency increases. The $E[V_T]$ in Figure 7(f) continues this pattern with the GBM producing the smallest losses and the Student t with $\nu = 7$ the largest.

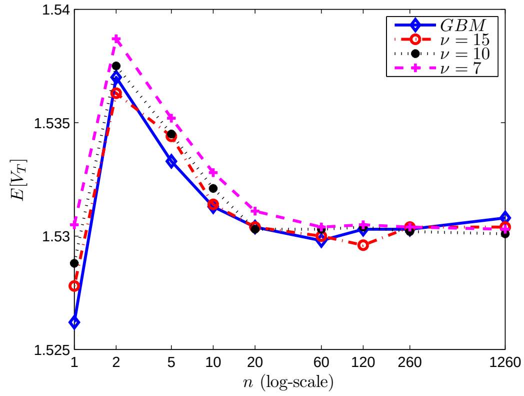
Overall these results indicate that kurtosis is not too crucial a factor for CPPI. However, from the results it can be suggested that $m = 3$ and monthly rebalancing are good parameters to minimise risk and achieve a satisfactory payoff. Although $m = 3$ shows larger losses in Figure 7(e), it has a very low probability of loss as shown in Figure 7(c).

3.7 Impact of Jump Size and Jump Frequency

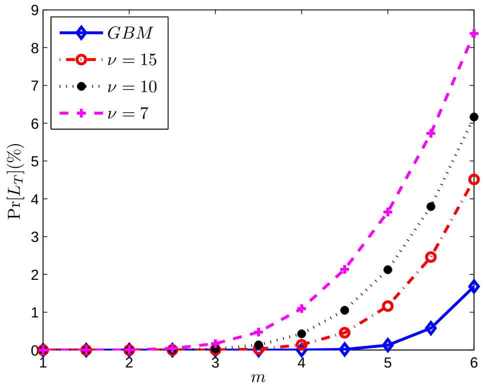
In order for the performance of the jump diffusion to be fairly compared to that of the GBM, the volatility of the process must be simulated with a volatility of 20% as used for the GBM. Two approaches have been taken when keeping the variance of the process in mind. Simulations have been performed with both $\sigma = 0.2$ and the standard deviation of the process (σ_j) equal to 0.2. In the latter, the σ was scaled (lowered) to accommodate the changes to the b and λ parameters according to Equation (17). The values used for the magnitude and frequency of jumps, b and λ, are based on empirical estimates for the FTSE 100 determined by Honore (1998). The σ and s superscripts on the b and λ in Figures 8 and 9 are explained as follows: the σ superscript indicates that σ in Equation (17) is fixed at 0.2 and the s superscript denotes that the standard deviation of the process (σ_j) is fixed at 0.2. Therefore σ denoted processes have a higher volatility



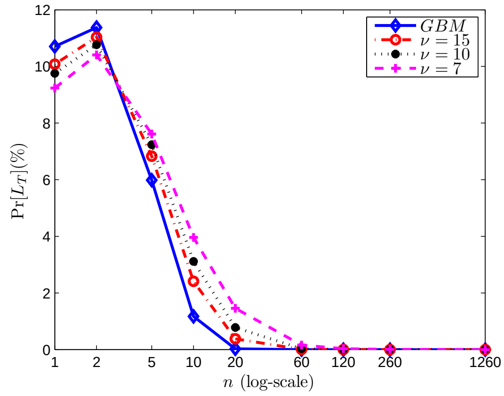
(a) Expected terminal portfolio value V_T



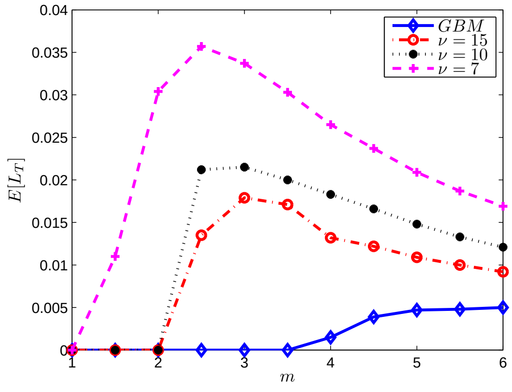
(b) Expected terminal portfolio value V_T



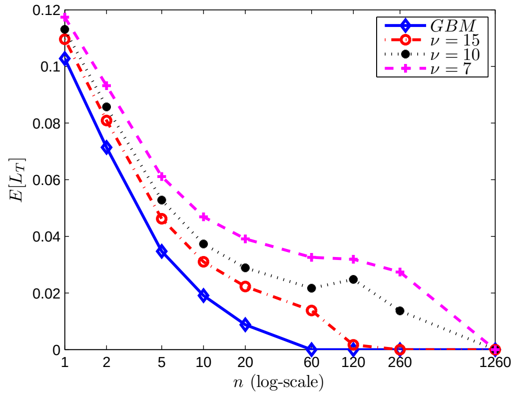
(c) Probability of losses L_T



(d) Probability of losses L_T



(e) Expected value of losses L_T



(f) Expected value of losses L_T

Figure 7: Comparison of GBM with log-t for different ν in conjunction with varying multiplier values m (left; $n = 60$) and rebalancing frequencies n (right; $m = 3$)

than their s counterparts. The jump process has been assigned a zero mean ($a = 0$) for all simulations.

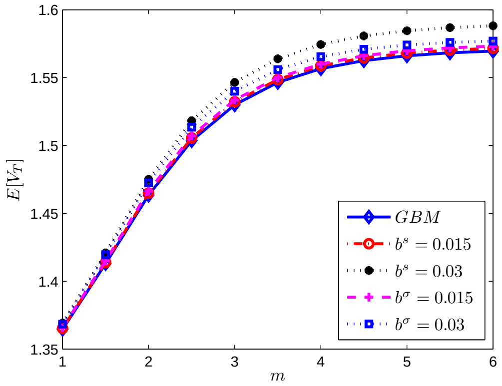
Figures 8(a), 8(c) and 8(e) compare the performance of the CPPI for the GBM against the jump diffusion for $b = 0.015$ and $b = 0.03$ with a range of m values. With reference to the $E[V_T]$ plots in Figure 8(a), the jump processes produce slightly higher payoffs than the GBM, with $b^s = 0.03$ producing the highest payoff. Since $b^s = 0.03$ dominates $b^\sigma = 0.03$ (and $b^s = 0.015$ dominates $b^\sigma = 0.015$) and b^s has the same volatility as the GBM it can be inferred that this positive effect on the payoff is the result of jumps. Figure 8(c) shows the probability of loss to be greatest for $b^\sigma = 0.03$, with $b^\sigma = 0.015$ producing a higher $\Pr[L_T]$ than $b^s = 0.015$. This indicates that the volatility of the process has more of a detrimental effect on the payoff than the jump size. In Figure 8(e) the processes with large jump sizes have larger losses. There is little to separate the b^σ and b^s processes other than discrepancies caused by Monte Carlo artifacts.

Figures 8(b), 8(d) and 8(f) show how different rebalancing frequencies effect the CPPI for $b = 0.015$ and $b = 0.03$. Although there appears to be a considerable difference between the plots for $E[V_T]$, the scale on the y-axis reveals that these are in fact relatively small, although $b^s = 0.03$ clearly produces the best payoff. The $E[L_T]$ also shows little difference between the processes, but the $\Pr[L_T]$ indicates that the $b^s = 0.03$ process has a lower probability of loss than the others. As the $b^s = 0.03$ process has a noticeably higher payoff and lower risk, it can be said that for $m = 3$ jumps have a positive impact. The jump processes appear to be effected by the change in rebalancing frequencies in the same way as GBM and the presence of jumps does not produce any significant new behaviour. Similar results for varying multiplier values are found for $\lambda = 1$ and $\lambda = 5$ as $b = 0.015$ and $b = 0.03$, with increases in λ producing higher payoffs and risk in comparison with GBM. For varying rebalancing frequencies there is also a similarity between the λ and b results, with the λ^σ processes (with the higher volatility) producing greater risk than the GBM. The plots for λ can be seen in Figure 9.

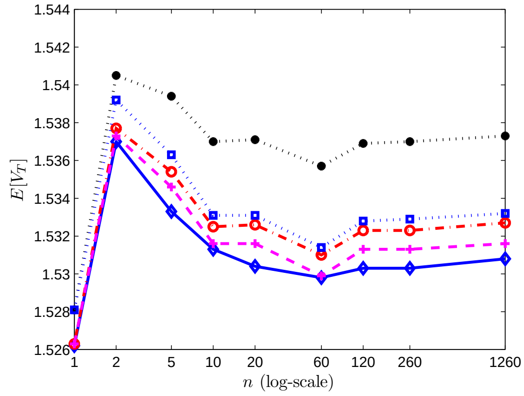
Overall the results from this section suggest that the impact of jumps for the parameters considered does not introduce any significant benefits or risk to the CPPI beyond that already analysed for the GBM. Certainly, the effect of volatility poses a much more substantial risk to the CPPI. Intuitively, jumps should be a key risk factor to the CPPI by their very nature of causing instantaneous movements in the risky asset which the CPPI cannot protect against. However, the size of the jumps is a key consideration in how the jumps will affect the risk of the CPPI. Equation (13) gives the amount by which the risky asset must drop by between rebalancing points before the floor is breached and a loss is realised. For monthly rebalancing and $m = 3$, this amount is approximately 33%. With jumps having a maximum volatility of 3% ($b = 0.03$) and zero mean, the likelihood of them adding significantly to the risk of shortfall is low. The volatility of the random walk part of the process (σ), is therefore the biggest contributor to the risk. Additionally, only symmetrical jumps are considered and it would be expected that if downward jumps were more likely than upwards jumps, then the risk and payoff would be more substantially affected.

3.8 Management Fees and Transaction Costs

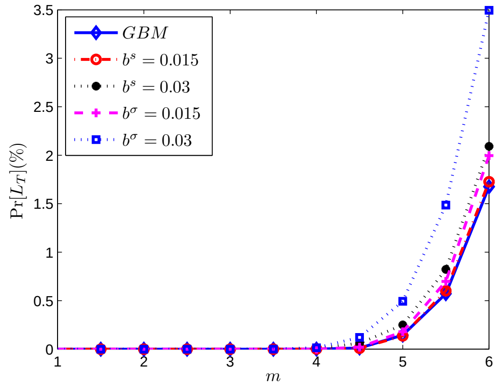
This section analyses the effects that management fees and transaction costs have on the performance of the CPPI. Figures 10(a) and 10(b) shows the effect that management fees



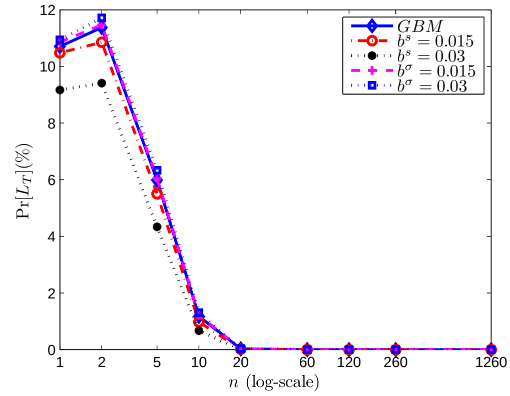
(a) Expected terminal portfolio value V_T



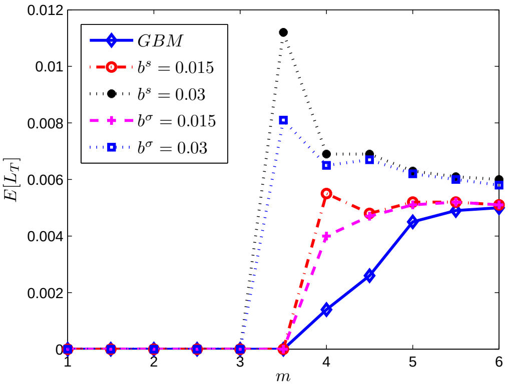
(b) Expected terminal portfolio value V_T



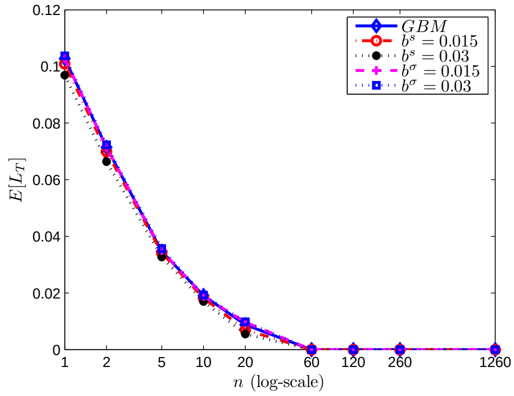
(c) Probability of losses L_T



(d) Probability of losses L_T

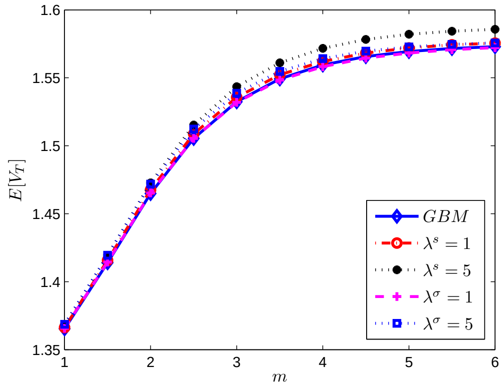


(e) Expected value of losses L_T

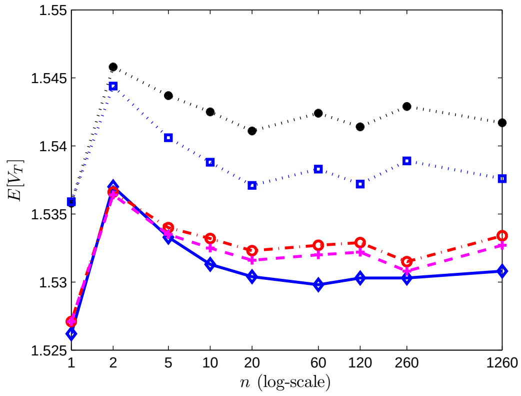


(f) Expected value of losses L_T

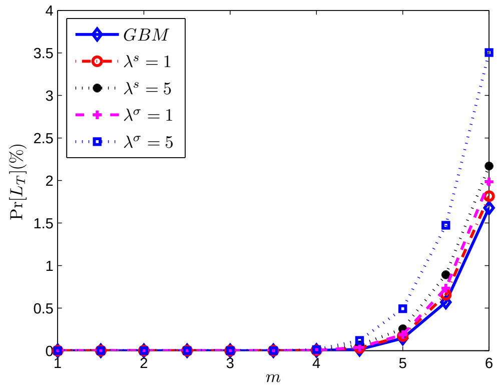
Figure 8: Comparison of GBM with jump diffusion for $\lambda = 5$ in conjunction with varying multiplier values m (left; $n = 60$) and rebalancing frequencies n (right; $m = 3$)



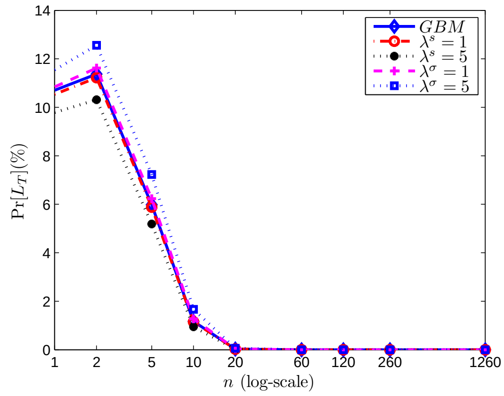
(a) Expected terminal portfolio value V_T



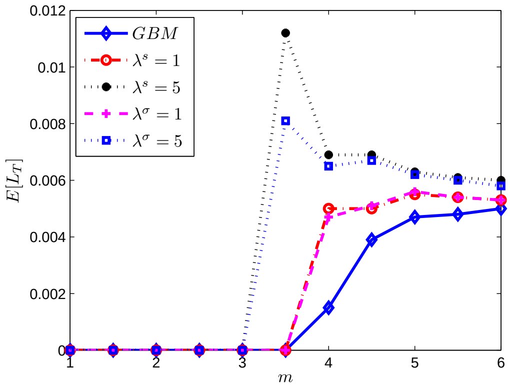
(b) Expected terminal portfolio value V_T



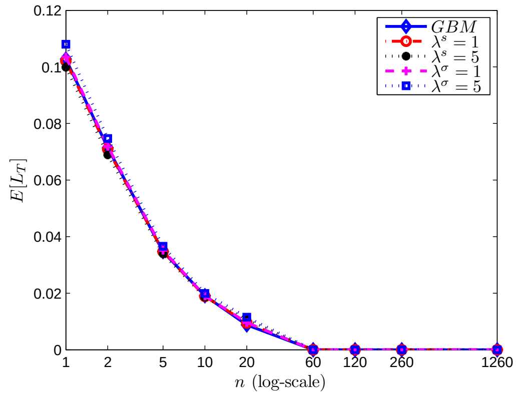
(c) Probability of losses L_T



(d) Probability of losses L_T

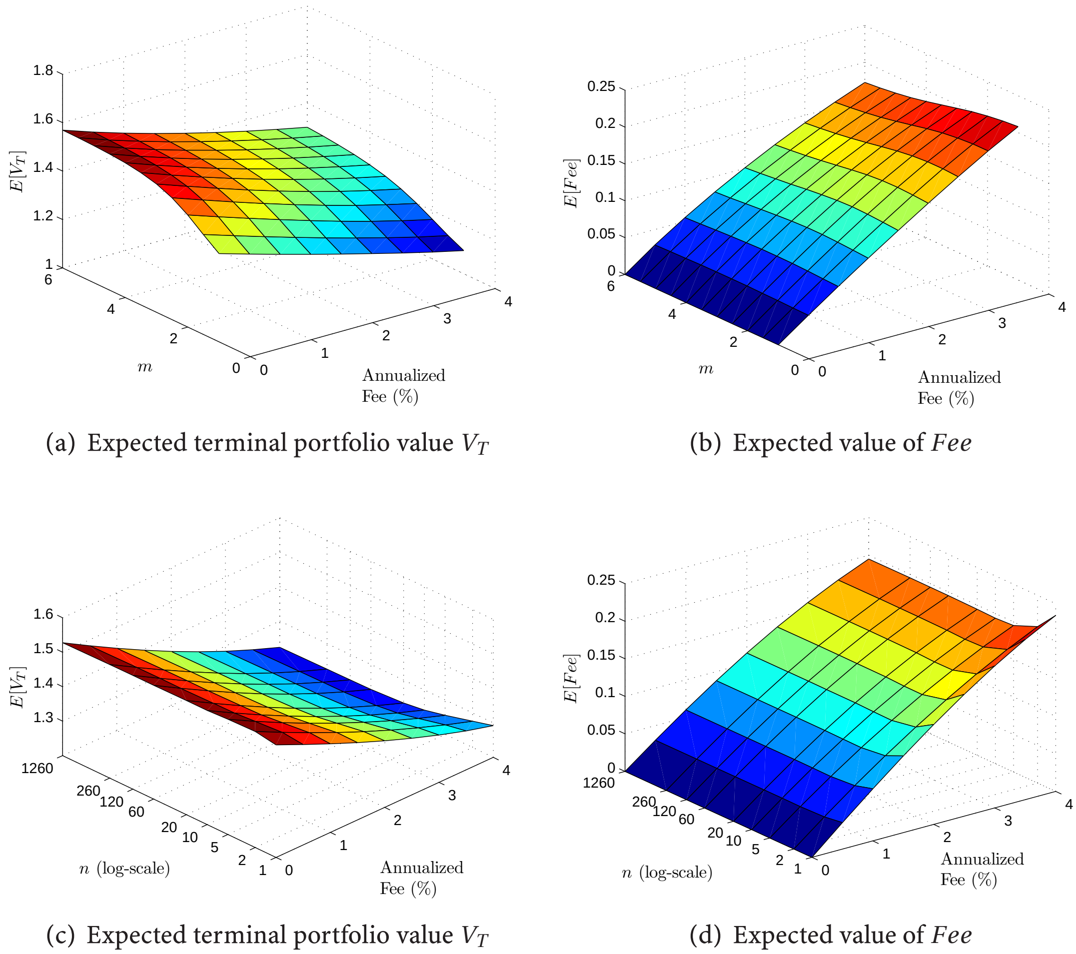


(e) Expected value of losses L_T



(f) Expected value of losses L_T

Figure 9: Comparison of GBM with jump diffusion for $b = 0.03$ in conjunction with varying multiplier values m (top; $n = 60$) and rebalancing frequencies n (bottom; $m = 3$)



(a) Expected terminal portfolio value V_T

(b) Expected value of *Fee*

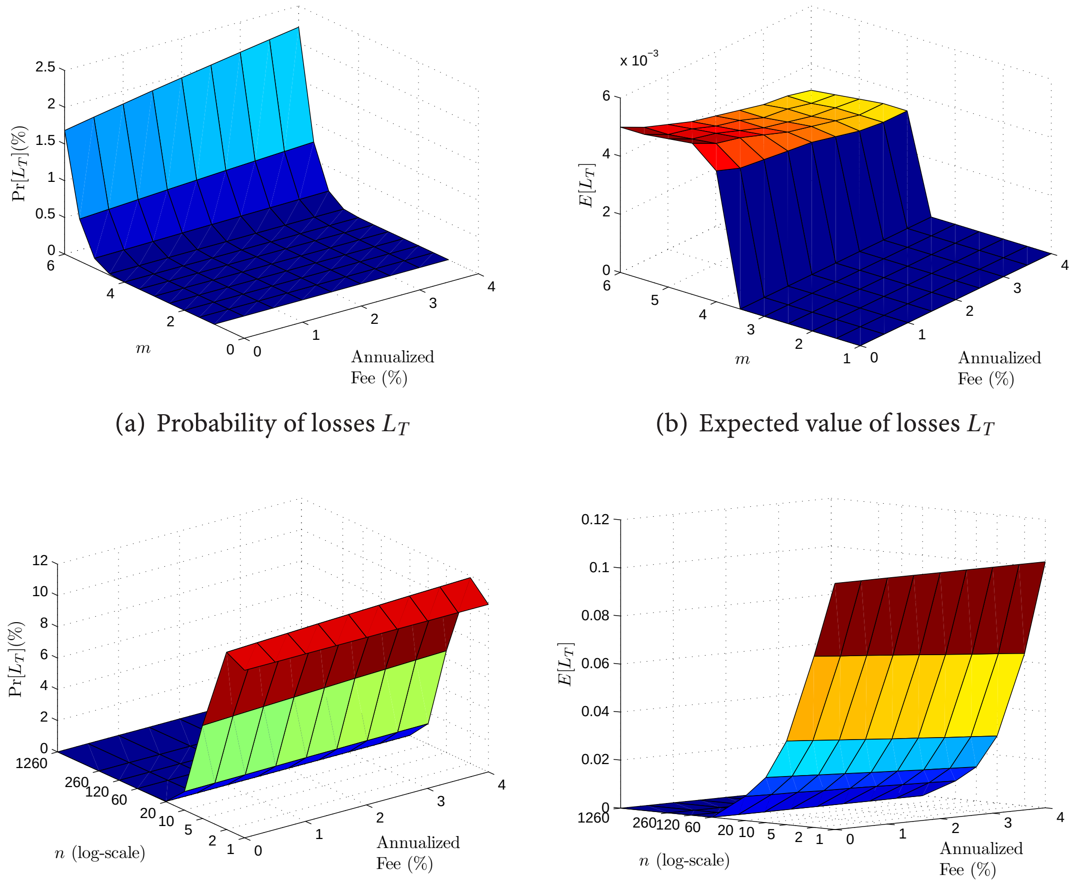
(c) Expected terminal portfolio value V_T

(d) Expected value of *Fee*

Figure 10: Effect of management fees in conjunction with varying multiplier values m (top; $n = 60$) and rebalancing frequencies n (bottom; $m = 3$)

(in the range of 0 to 6%) have on the CPPI in relation to varying multiplier values. As the fee increases $E[V_T]$ falls by a considerable amount, which is more profound for lower m values. The loss graphs indicate the incidence and size of losses is not greatly affected by the fees. In particular the expected value of losses are small and not very significant e.g. the highest loss incurred is approximately 0.5% of the notional (see Figure 11(b)). Figure 10(b) shows the expected value of the capital taken in fees over the investment horizon. It can be seen that more than 10% of the notional is taken as a fee on average for $m = 3$ when the management fee is 2% per annum. Although it may be expected that higher values of m would generate higher value portfolios and therefore greater fees, this does not appear to be the case. The implementation of the fee charging rule (see Equation (22)) is likely to be the cause of this. Since no fee is charged if it will result in a shortfall, then the amount in fees taken balances out for low values of m with few likely shortfalls and high values of m with many potential shortfalls.

Figures 10(c) and 10(d) illustrate the impact of management fees for different rebalancing frequencies. The payoff declines as more frequent trading occurs and this is more pronounced for higher fees. This can be explained because more frequent trading results in fees being taken more often (calculated pro rata using the annualised fee) and



(a) Probability of losses L_T

(b) Expected value of losses L_T

(c) Probability of losses L_T

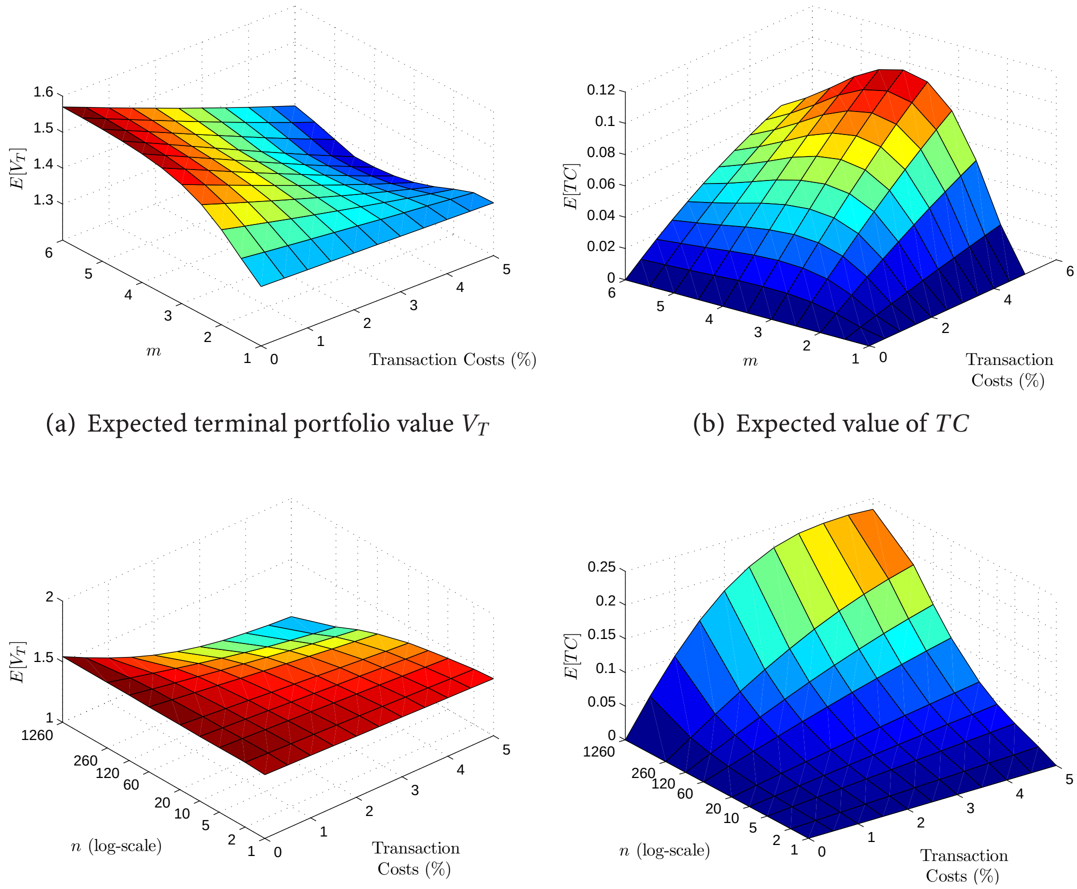
(d) Expected value of losses L_T

Figure 11: Effect of management fees in conjunction with varying multiplier values m (top; $n = 60$) and rebalancing frequencies n (bottom; $m = 3$)

this inhibits the growth of the portfolio while not affecting the risk (see Figures 11(c) and 11(d)). The portfolio does not get a chance to grow substantially in a shorter period and taking the fee reduces the free capital available for further investment in the risky asset. Since this effect does not occur for lower trading frequencies and does not affect the amount in fees taken, it would be more beneficial to deduct fees annually regardless of the portfolio rebalancing frequency. The result should be the same amount taken in fees, but without reducing the payoff for higher trading frequencies.

Finally, it should be noted that the expected value of the amount taken in fees greatly out-weighs the expected value of the amount lost due to shortfall. From the seller's perspective this is a good thing, but the buyer may question whether the fees are excessive.

Figures 12(a) and 12(b) show how varying rates of transaction costs affect the CPPI in conjunction with different m values. For $m \geqslant 1$ the payoff can be seen to be decreasing as the rate of transaction costs increases. For higher costs, around 4 to 5%, there is a clear dip in $E[V_T]$ between $m = 2$ and $m = 5$. The reason for this is that these values of m are more likely to result in larger changes in the portfolio when rebalancing occurs and so incur greater costs. Figure 12(b) supports this, showing a peak in the amount of transaction costs paid around theses same values of m for higher transaction costs.



(a) Expected terminal portfolio value V_T

(b) Expected value of TC

(c) Expected terminal portfolio value V_T

(d) Expected value of TC

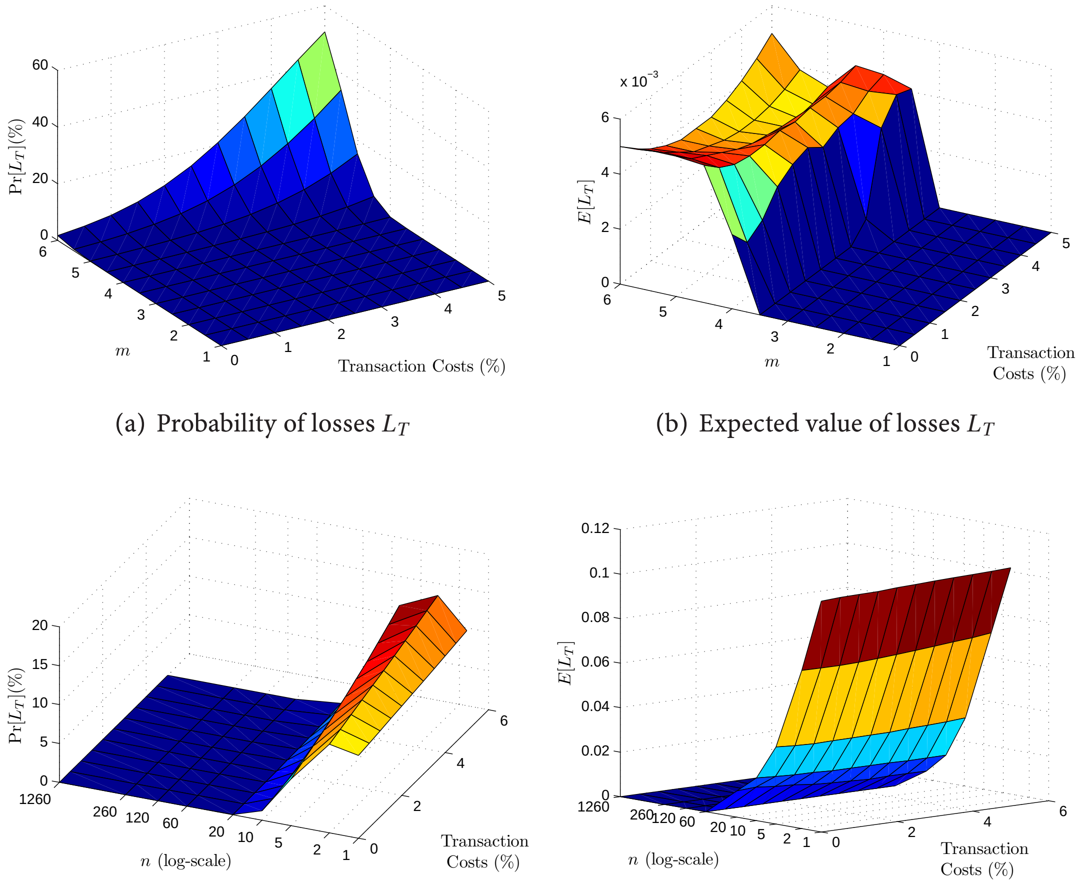
Figure 12: Effect of transaction costs in conjunction with varying multiplier values m (top; $n = 60$) and rebalancing frequencies n (bottom; $m = 3$)

Looking at transaction costs with varying rebalancing frequencies, Figure 12(c) shows the $E[V_T]$ drops considerably as n and the rate of transaction costs increases. This is expected since the more frequently the portfolio is rebalanced, the greater the amount of trading and hence the higher the costs (see Figure 12(d)).

Transaction costs do not appear to affect the risk of CPPI when $m = 3$, but do increase the probability of loss for $m \geqslant 4$ (see Figures 13(a) and 13(b)). In these cases the high amount of costs paid severely inhibit the CPPI's ability to stay above the floor. Clearly the less frequently the portfolio is rebalanced the less the transaction costs and the higher the $E[V_T]$. However, when considering the performance overall, monthly rebalancing again offers a good tradeoff between risk, costs and payoff.

3.9 CPPI versus Riskless and Gapless Portfolios

To a buyer seeking an investment that is guaranteed to protect at least their initial capital, there exist a couple of investments other than the standard CPPI ($m > 1$), which can be considered for comparison. The riskless portfolio, comprising of a 100% investment in the risk-free asset will always pay more than the initial investment at maturity. The



(a) Probability of losses L_T

(b) Expected value of losses L_T

(c) Probability of losses L_T

(d) Expected value of losses L_T

Figure 13: Effect of transaction costs in conjunction with varying multiplier values m (top; $n = 60$) and rebalancing frequencies n (bottom; $m = 3$)

Table 1: Mean and median ratios under GBM, $m = 3$. V_T^c, V_T^{rf} and $V_T^{m=1}$ denote the terminal value of the Standard CPPI, the riskless and the gapless portfolio, respectively.

		No Fee		1.5% Fee		100% Max. Leverage	
	σ	V_T^c/V_T^{rf}	$V_T^c/V_T^{m=1}$	V_T^c/V_T^{rf}	$V_T^c/V_T^{m=1}$	V_T^c/V_T^{rf}	$V_T^c/V_T^{m=1}$
mean	0.1	1.2193	1.1373	1.1138	1.0393	1.2453	1.1582
	0.2	1.1918	1.0878	1.0904	0.9978	1.2390	1.1123
	0.3	1.1670	1.0365	1.0749	0.9619	1.2265	1.0415
	0.4	1.1467	0.9916	1.0672	0.9365	1.2086	0.9653
	0.5	1.1314	0.9588	1.0648	0.9215	1.1991	0.9065
	0.6	1.1218	0.9399	1.0658	0.9151	1.1931	0.8736
median	0.1	1.1630	1.1015	1.0515	0.9960	1.1533	1.0923
	0.2	0.9850	0.9505	0.9009	0.8798	0.9688	0.9356
	0.3	0.8459	0.8837	0.7945	0.8539	0.8358	0.8694
	0.4	0.7900	0.8743	0.7790	0.8583	0.7873	0.8463
	0.5	0.7793	0.8833	0.7788	0.8751	0.7791	0.8531
	0.6	0.7788	0.9015	0.7788	0.8972	0.7788	0.8777

gapless portfolio, which is the same as a CPPI with $m = 1$ is essentially a buy-and-hold strategy that guarantees at least the initial capital.

Table 1 shows the average CPPI payoff ratios for an $m = 3$ portfolio in comparison to riskless (V_T^{rf}) and gapless ($V_T^{m=1}$) portfolios taking the mean and median respectively. Payoffs are observed from the client's (buyer's) perspective, with at least the guaranteed amount being paid at maturity (see Equation (29)). The median value of the ratios has been considered in addition to the mean because it ignores the magnitude of outliers that skew the distribution and increase the mean. The median therefore gives a better indication of what the payoff will be for a *one shot* investment rather than the asymptotic payoff.

In the upper part of Table 1 it can be seen that as the volatility increases the ratios drop showing that the CPPI becomes a poorer investment in relation to the other two portfolios for higher volatility. Of particular concern is that with a volatility of 40% (and higher) the $m = 3$ CPPI fails to outperform the $m = 1$ CPPI even in the absence of fees. When a realistic fee of 1.5% is considered the $m = 3$ CPPI fares even worse. In terms of median values (lower part of Table 1), without fees the CPPI performs significantly worse than the gapless portfolio at a volatility of just 20% and even fails to beat the riskless portfolio. When the CPPI has a maximum leverage of 100% the CPPI gains a significant payoff increase over the riskless and gapless portfolios for $\sigma = 0.1$. Increasing the to $\sigma = 0.2$, as with the unlevered CPPI, Table 1 shows that the levered portfolio compares poorly. The results suggest that in periods of high volatility the investor would be best suited to consider an investment in the risk-free or gapless portfolios.

Table 2: Statistics for cumulated returns and conditional losses

(a) Standard CPPI

	$\ln(V_T)$				$\ln(V_T^L)$			
m	Mean	Std	Skew	Kurt	Mean	Std	$\Pr[L_T](\%)$	$E[E_T/V_T](\%)$
1	0.3036	0.1179	0.9808	4.5313	–	–	–	25.69
2	0.3437	0.2553	1.4844	5.5514	–	–	–	52.08
3	0.3605	0.3372	1.2029	3.9112	–	–	–	61.36
4	0.3644	0.3718	1.0373	3.3226	−0.0082	0.0010	0.00	62.18
5	0.3644	0.3876	0.9542	3.0724	−0.0054	0.0081	0.14	61.15
6	0.3633	0.3959	0.9073	2.9410	−0.0051	0.0084	1.69	59.73

(b) Standard CPPI with 100% maximum leverage

	$\ln(V_T)$				$\ln(V_T^L)$			
m	Mean	Std	Skew	Kurt	Mean	Std	$\Pr[L_T](\%)$	$E[E_T/V_T](\%)$
1	0.3037	0.1179	0.9797	4.5000	–	–	–	25.70
2	0.3438	0.2602	1.7170	7.2525	–	–	–	53.95
3	0.3584	0.3942	2.1168	8.4687	–	–	–	75.40
4	0.3543	0.4830	2.0478	7.2716	−0.0042	0.0021	0.00	80.67
5	0.3442	0.5323	1.9746	6.6087	−0.0096	0.0149	0.23	77.28
6	0.3330	0.5601	1.9470	6.3262	−0.0104	0.0184	3.10	71.31

3.10 Distribution of Returns and Losses

Table 2(a) shows the log-moments of both the terminal values of all portfolios and the terminal values of only those portfolios that experienced losses, denoted V_T^L (where $V_T^L = V_T|V_T < G$). The expected value of the log-payoff of V_T can be seen to be peaked at $m = 4$, $m = 5$ with a value of 0.3644. These values of m can be considered to perform well for the chosen parameters of the simulation (GBM price process, monthly trading, $r = 0.05$, $\mu = 0.1$, $\sigma = 0.2$). Lower ms do not allow enough exposure to the risky asset to benefit more from its growth, while higher ms are too greatly affected by the increase in the number of losses. The reader may be aware that multiplier values of 4 and 5 are quite high for a 5 year CPPI product. Indeed they result in 88% and 100% of the initial capital being allocated to the risky asset.[2] However, looking at the final column of Table 2(a) it is known that on average for $m = 4$ and $m = 5$ respectively the terminal risky allocations will be about 62% and 61%. This suggests, at least for a maturity of 5 years, that it is better to invest a large amount of the capital in the risky asset initially and rely on the CPPI strategy to reduce exposure over time. However starting with an exposure much more than the leverage limit (i.e. for $m > 5$) is detrimental to the CPPI's performance.

Table 2(a) shows the distribution of CPPI payoffs exhibiting positive skewness despite the underlying risky asset following a GBM. This proves that the CPPI is effective in reducing losses, while preserving much of the upside.

[2] $(1-\exp(-0.05\cdot5))\cdot4\cdot100\% = 88.48\%, (1-\exp(-0.05\cdot5))\cdot5\cdot100\% = 110.60\%$ but the leverage constraint caps this at 100%. See Equations (1), (2), (4) and (19).

Table 2(b) gives the moments of a 100% levered CPPI and provides evidence of large outliers shifting the mean. When compared to the unlevered CPPI in Table 2(a), it can be seen that the levered CPPI is heavily skewed and exhibits significant extra kurtosis. Additionally, $E[\ln(V_T)]$ is lower for the levered portfolio than the unlevered, which is not what is seen for $E[V_T]$ in Figure 5(a). Thus the impact of large outliers on the mean is apparent.

4 Conclusion

This paper investigates the statistical properties and main issues in the implementation of the CPPI strategy. In particular, the results show that increased volatility is detrimental to the CPPI's performance. When volatility is high, the implementor can limit risk by reducing m and increasing the trading frequency. In fact increasing the number of rebalances would be the preferred approach (in the absence of transaction costs), as it would reduce risk without having as big a negative effect on the payoff as reducing m. Overall the results indicate that for a CPPI under discrete trading with typical market assumptions, a good multiplier value lies between 3 and 5 and a suitable rebalancing frequency is monthly. These values give a good tradeoff between risk, payoff and costs.

Excess kurtosis in the risky asset (i.e. beyond that of the GBM) does little to affect the CPPI. This is at least true for a symmetrical distribution, but unlikely to be the case for skewed distributions. Jumps in the price of the risky asset do have a noticeable affect on the CPPI, with increased jump size and frequency having a positive impact on the payoff. However, for normally distributed jumps and the values of the parameters considered the impact is not substantial. In particular, volatility is a bigger contributor to the risk of the CPPI than the size and frequency of the jumps themselves.

The gapless portfolio ($m = 1$) is a good option for investors looking to have some risky exposure while retaining the majority of the capital in the risk-free asset. This is even more true when the underlying is particularly volatile. Additionally, the gapless portfolio has the benefit of little or no management fees or transaction costs. However, if the investor wishes to utilise increased leverage, which has been shown to produce increased payoffs, then the gapless portfolio will be unable to accommodate this. Although the results show the CPPI performing poorly compared to the gapless portfolio for volatilities of the underlying greater than 10%, it should be noted that the traits of the underlying are determined by its type. Hedge funds typically have a volatility of less than 10% and so their use in the CPPI could provide an attractive investment.

Management fees have a significant impact on the payoff of the CPPI. Even with relatively low volatilities, the payoff is substantially eroded. It has been shown that taking management fees from the CPPI at more frequent intervals impedes its growth. By taking fees less frequently the expected payoff of the CPPI is increased, while the amount taken in fees remains the same. Considering also that is has been shown that in many cases it is better for a buyer of the CPPI to invest in a gapless portfolio, the addition of fees to an already poorly performing portfolio further reduces its attractiveness as an investment. It is therefore important for an investor to consider whether they believe that the fees justify the returns they expect to receive. Furthermore, the buyer of the product must consider their risk aversion and whether the potential increased performance of the CPPI makes it more favourable than a purely risk-free or gap-free investment. From the buyer's perspective, the charging of management fees should be a reflection of seller's skills to manage the portfolio in such a way as to produce excess returns.

The CPPI essentially allows the implementor to choose only two components: the multiplier value and the rebalancing frequency. The results demonstrate that these two parameters have a significant impact on the performance of the CPPI and being able to optimize their values would be very beneficial to the CPPI. However, the values for these parameters are often fixed for the entire life of the investment. Suggested further work includes dynamically optimizing the value of m as well as deciding when to rebalance in reaction to market conditions.

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A Derivation of Continuous Time CPPI under a Lognormal Price Process

This appendix provides the derivation of the standard CPPI value together with it's first and second moments. The value of the floor growing at the constant interest rate r can be determined at time t as given in Equation (2) by

$$F_t = e^{-r(T-t)}G.$$

The dynamics of the risky asset are assumed to follow a standard geometric Brownian Motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where μ is the expected rate of return , σ the volatility and W_t the standard Wiener process. The solution to the above stochastic differential equation (s.d.e.) is

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \tag{30}$$

where S_0 is the initial value of the risky asset. The expected value and variance of the risky asset is (see (Hull, 2005, p.282))

$$E[S_t] = S_0 \exp\{\mu t\}$$
$$Var[S_t] = S_0^2 \exp\{2\mu t\}\left(\exp\{\sigma^2 t\} - 1\right).$$

The cushion is the difference between the portfolio value and floor (from Equation (1))

$$C_t = V_t - F_t.$$

The exposure to the risky asset is defined as a multiple m of the cushion (from Equation (4))

$$E_t = mC_t.$$

At any time t, the change in the value of the CPPI portfolio is given by

$$dV_t = (V_t - E_t)\frac{dB_t}{B_t} + E_t\frac{dS_t}{S_t},$$

where B_t is the risk-free asset which follows the same dynamics as the floor. The cushion then satisfies the following

$$\begin{aligned} dC_t &= dV_t - dF_t = (V_t - E_t)\frac{dB_t}{B_t} + E_t\frac{dS_t}{S_t} - dF_t \\ dC_t &= (C_t + F_t - mC_t)\frac{dB_t}{B_t} + mC_t(\mu dt + \sigma dW_t) - dF_t \\ dC_t &= C_t[((1-m)r + m\mu)dt + m\sigma dW_t]. \end{aligned}$$

Rearranging the s.d.e. satisfied by the cushion gives

$$\frac{dC_t}{C_t} = \mu_c dt + \sigma_c dW_t,$$

where

$$\begin{aligned} \mu_c &= (1-m)r + m\mu \\ \sigma_c &= m\sigma \end{aligned}$$

Hence, the cushion follows a standard GBM and therefore

$$C_t = C_0 \exp\left\{\left(\mu_c - \frac{1}{2}\sigma_c^2\right)t + \sigma_c W_t\right\}. \tag{31}$$

From Equation (30) it follows that

$$W_t = \frac{1}{\sigma}\left[\ln\left(\frac{S_t}{S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)t\right]$$

and substituting this result into Equation (31) gives

$$\begin{aligned} C_t &= C_0\left(\frac{S_t}{S_0}\right)^m \exp\left\{\left(r - m\left(r - \frac{\sigma^2}{2}\right) - \frac{1}{2}m^2\sigma^2\right)t\right\} \\ V_t &= F_t + (V_0 - F_0)\left(\frac{S_t}{S_0}\right)^m \exp\left\{\left(r - m\left(r - \frac{\sigma^2}{2}\right) - \frac{1}{2}m^2\sigma^2\right)t\right\}. \end{aligned}$$

The expected value and variance of V_t can be deduced from the fact that the cushion C_t follows a GBM (Equation (31)). Hence the expected value is

$$\begin{aligned} E[V_t - F_t] &= C_0 \exp\{\mu_c t\} = C_0 \exp\{((1-m)r + m\mu)t\} \\ E[V_t] &= F_t + (V_0 - F_0)\exp\{((1-m)r + m\mu)t\} \end{aligned} \tag{32}$$

and the variance

$$\begin{aligned} Var[V_t - F_t] &= Var[C_t] = C_0^2 \exp\{\mu_c t\}(\exp\{\sigma_c^2 t\} - 1) \\ Var[V_t] &= (V_0 - F_0)^2 \exp\{((1-m)r + m\mu)t\}\left(\exp\{m^2\sigma^2 t\} - 1\right). \end{aligned} \tag{33}$$