Optimal Level of Leverage using Numerical Methods

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Mar 05, 2012

Abstract

Advances in computer software and in agent-based computational finance allow the use of numerical methods in order to optimize different type of functions. In this paper we propose a numerical method in order to find the level of leverage that maximizes the geometric mean of a series of historical daily returns. One of the advantages of the use of numerical methods is that it requires less assumption than analytical methods with closed-form solutions. Furthermore, after running an experiment, the analysis of the results demonstrates that the use of numerical methods yields a higher geometric mean than the use of analytical method based on the assumption of a geometric Brownian motion.

JEL Classification: C61
Keywords: Optimal level of Leverage, geometric Brownian motion, numerical methods.

1 Introduction

Today, leverage plays a very important role in the financial markets. New instruments designed to facilitate leveraged investments such as Financial Spread Betting (FSB) and Contract for Difference (CFD) in conjunction with technology advances in electronic markets make leverage accessible for every type of investor, from small individual investors up to big funds. For example, in 2007, 30% of the volume on the London Stock Exchange was driven for leveraged investments (FSA 2007).

However, to the best of our knowledge, there are relatively few studies linking leverage and agent-based computational finance (ABCF). The existing papers analyse the implications of leverage on the financial markets (Geanakoplos 2009; Thurner, Farmer, and Geanakoplos 2010). ABCF provides tools to test hypotheses regarding different aspects of finance. There is a considerable amount of research on trading systems using ABCF tools (Lo and MacKinlay 1990; LeBaron 2002; LeBaron 2006; LeBaron 2000; Martinez-Jaramillo and Tsang 2009; Iori and Chiarella 2002), but the effects of leverage on capital returns have been receiving relatively little attention in these models. It is commonly understood that if an unleveraged trading system is good and yields superior returns, consequently, leverage will only increase such returns proportionally and no further analysis is necessary to comprehend such results because it is intuitively obvious. However, this common understanding is incorrect; recent studies about leverage demonstrate that excessive levels of leverage are considered irrational behaviour (Geanakoplos 2009; Thurner, Farmer and Geanakoplos 2010) which demonstrates that studies linking leverage, trading system and ABCF are relevant.

Leverage has the ability to improve the investment performance, but, if used to an excessive level, leverage can lead the investor to ruin. Thus, if an excessive level of leverage is bad for investments; could leverage be good on some level? Is there an optimal level of leverage? If yes, could it be quantified?

Assuming that the investor objective is to maximize the end of period wealth, Kelly (1956) demonstrates that on binomial games with positive expectation, there is one specific value of leverage that maximizes the geometric mean or the end of period wealth. He proposed a model to obtain this value and this model was called Kelly criterion. Today, one of the possible uses of Kelly criterion is to determine the optimal level of leverage on investments considering the possible returns on different discrete scenarios. However, the Kelly criterion cannot be properly used to determine the optimal level of leverage on continuous games where the possible outcomes are unlimited under certain distribution, like investments on financial markets.

In order to relax this restriction, Rotando and Thorp (1992) revise the Kelly criterion version for continuous gambling games. In this case, the issue is that the use of Kelly criterion for continuous gambling games is complicated. Continuous games assume infinite number of outcomes and the Kelly criterion was designed to assume limited number of outcomes. Hence, in order to limit the number of outcomes and respect the Kelly criterion assumptions, the authors assumed that the returns in the financial market follow a normal distribution and such distribution should be transformed to quasi-normal distribution by cutting off the tails. It allows the use of Kelly criterion on continuous games, however, it still has an important issue, namely the assumptions that the returns follow a normal distribution.

In order to obtain a more realistic approach, Peters (2009) introduces a new model to obtain the optimal level of leverage. He assumes that asset prices follow a geometric Brownian motion and demonstrates that the optimal level of leverage is simply the estimation of return divided by its variance. The point in this proposition is the assumptions on the price process. In the literature, there is a considerable discussion on the price process and return characteristics (Cont 2001).

In this paper, in order to avoid any discussion about the return characteristics, we propose to estimate the optimal level of leverage using a numerical non-parametric method. In our case, we use a line-search method to get the level of leverage that maximizes the geometric mean of a series of historical returns. The insight for this proposition is that analytical methods proposed in the literature are not able to capture the presence of the fat tails on the return distribution (Cont 2001) and consequently, their results could not represent the real optimal level of leverage.

In order to test the hypothesis that numerical methods yield superior geometric mean to analytical methods, we calculated the optimal level of leverage using geometric Brownian motion and using numerical methods of 10^4 series of daily return of the components of Dow Jones index. In the experiment, the stock, the length of the series and the period are randomly selected, and the result is that the numerical method shows superior level of leverage and higher geometric mean to a closed-form model based on geometric Brownian motion.

This paper is structured as follows. In section 2, we present the general aspects of leverage. In section 3, we review the literature about the different methods to measure and to estimate the investment performance. In section 4, we detail the models to optimize the level of leverage proposed in the literature. In section 5, we present the results of the experiment that compares two different methods to calculate the optimal level of leverage, geometric Brownian motion and numerical methods. Finally, in section 6, we conclude by presenting some final remarks and possible future work.

2 The Growth Optimal Portfolio

Studies of leverage are entirely associated to debates of individual investor objectives. In the literature, the investor objectives can be divided in two different approaches: mean-variance and growth optimal portfolio. Leverage is linear in the mean-variance approach, and concave in the growth optimal portfolio approach. Due to linearity of leverage, studies of leverage are irrelevant in the mean-variance approach, because even when leverage is introduced into the parameters of the model, it does not alter the model itself, i.e., the model is independent of leverage. However, due to the non-linearity of leverage in the growth optimal portfolio approach, the study of leverage is important because the introduction of leverage on the parameters alters the results of the model.

There is a very important debate about the objectives of individual investors. While Markowtiz (1952) develops a mean-variance method to determine the optimal portfolio to the next period; Kelly (1956) and Latan (1959) argue that the main objective of the individual investor is to maximize the end of period wealth; hence, the investor should be concerned about the capital rate of growth which could be measured using the geometric mean.

This debate continued during the 1960s and 1970s. Breiman (1961) argues that on a long sequence of trials the game objectives are to minimize the time to reach the target level of wealth, or to maximize the value of the end of the period wealth. Breiman (1961) and Hakansson (1971a; 1971b) show that the optimal strategy to attain both objectives is to maximize the expected value of the log of the terminal wealth which is the same as maximizing the geometric mean of return.

On the other hand, Samuelson (1971; 1979) argues that geometric mean maximization was only one among others investment rules and there is no sense on the belief on its superior results. He shows that the geometric mean rule leads to sub-optimal expected utility and, because the end of period expected is the sum of the utility for each period, the end of period wealth under geometric mean rule will also be sub-optimal.

The debate was theoretical with each author advocating different rules by using even more mathematical complex models. The reason is that during that period, the use of numerical methods was extremely difficult because of the low level of the computers technology compared to today's computers. The lack of technology obstructed the debate for more than 20 years. However, during the 1990s and the 2000s, new authors such as MacLean, Ziemba, and Blazenko (1992), Ross (1999), Hunt (2002), Leippold, Trojani, and Vanini (2004), Christensen (2005) and Estrada (2009), restarted the debate, but at this time, using numerical methods to analyse, simulate and compare different rules similar to the theory proposed on the 1970s.

Debates about investor objectives are associated with debates about the model to measure the investment performance, risk-return or geometric mean. The mean-variance approach is associated with the risk-return formula, and growth optimal portfolio is associated with the geometric mean formula. The risk-return formula is:

$$RR = \frac{\mu}{\sigma} \tag{1}$$

where RR is the risk-return and μ and σ are the arithmetic average and the standard deviation of the historical returns, respectively.

The alternative methodology to calculate the return is the geometric mean formula:

$$GM = \prod_{t=1}^{n}(1 + r_t)^{\frac{1}{n}} - 1 \tag{2}$$

Applying Taylor expansion, assuming returns Normally distributed and simplifying the Eq. (2) (See the appendix):

$$GM = (\mu - \frac{\sigma^2}{2}) \tag{3}$$

where GM is the geometric mean and μ and σ^2 are the arithmetic average and the variance of the historical returns.

A debate about the ideal methodology to measure average return has its origin in Williams (1936), who demonstrates that speculators, in a multi-period framework, should be concerned about the geometric mean instead of the arithmetic mean. The main argument is that risk-return does not consider the non-linearity of returns present on exponential compound series, like financial series, and such characteristic is only captured by the geometric mean (Latan 1959).

In order to introduce leverage, the Eq. (1) can be rearranged, and consequently the leveraged risk-return formula (RRL) is:

$$RRL = \frac{l\mu_i}{l\sigma_i} = RR \tag{4}$$

where l is the level of leverage; and rearranging the Eq. (2), the geometric mean formula (GM) is:

$$GM = (l\mu - l^2\frac{\sigma^2}{2}) = l(\mu - l\frac{\sigma^2}{2}) \tag{5}$$

From the Eq. (4) and Eq. (5), note that leverage is monotonic and linear in the risk-return formula, but is neither monotonic nor linear in the geometric mean formula. For this reason, we argue that the geometric mean formula is the appropriate model to measure the investment performance whether the investor uses leverage. Thus, the geometric mean formula should be used to calculate the optimal level of leverage.

3 Optimal Leverage Models

3.1 The Kelly Criterion

The importance of studies on leverage is not recent. Kelly (1956) demonstrates that there is an optimal level of leverage for binomial games such as coin tossing games. He assumes that the main objective of the player is to maximize the trial's expected value and, consequently, the end of period cumulative return. He also demonstrates that if you have a positive trial's expected value, the size of your leverage depends upon two factors:

the value of your expected value and the probability of ruin. This model is called the Kelly criterion. Originally, the Kelly criterion consists of a model to maximize returns on a binomial game similar to coin tossing. The model is described below.

Suppose that on each trial the win probability is $p > 1/2$ and, consequently, the probability of lose is $q = 1 - p < 1/2$. Once the outcome probability is defined, the question is to decide the amount of capital B_i to bet on each trial with the objective to maximize the expected value of the end of trials wealth, $E(X_n)$. Letting $T_i = 1$ if the i_{th} trial is a win and $T_i = -1$, if the i_{th} trial is a loss. Furthermore, letting X_0 is the initial capital, then $X_i = X_i - 1 + T_i B_i$, for $i = 1, ..., n$ and, consequently $X_n = X_0 + \sum_{i=1}^{n} T_i B_i$, then

$$E(X_n) = X_0 + \sum_{i=1}^{n} (p - q) E(B_i) > 0 \tag{6}$$

Assuming $p > 1/2 > q$ and, consequently the expected value of the end of trials wealth $E(X_n)$ is positive, the objective is to find the amount of the available capital which should be bet, B_i, and the amount of capital that should be saved for later trials, $X_i - 1$. If the player decides to bet all the capital, it increases the probability of ruin. On the other hand, if deciding to bet the minimal capital, the player reduces the probability to maximize the value of the end of trial's wealth. Therefore, there is some optimal level of leverage or optimal fraction, l, which balances the objective to maximize the expected value with the restriction to minimize the probability of ruin.

In the coin-tossing game, since the gambling probability and the payoff at each trial are the same, it is clear that the optimal level of leverage is the same for all trials. This assumption of fixed leverage helps us to comprehend the Kelly purpose. Maximizing the expected end of trial wealth is similar maximizing the expected value of the growth rate coefficient or the geometric mean, GM(l), where after some algebraic calculations:

$$GM_k(l) = E[\log \frac{X_n}{X_0}]^{\frac{1}{n}} = p\log(1+l) + q\log(1-l) \tag{7}$$

$$GM_k'(l) = \frac{p}{1+l} - \frac{q}{1-l} \tag{8}$$

$$GM_k''(l) = \frac{-l^2 + 2l(p-q) - 1}{(1-l^2)^2} \tag{9}$$

From the Eq. (9), it can be viewed that the function $GM_k(l)$ is concave on l, hence it is able to be maximized. Furthermore, solving the Eq. (8), the result is that the optimal level of leverage is $l^* = p - q$.

3.2 Rotando and Thorp Model

The Kelly criterion cannot be directly used on investments. The main reason is that with games like coin tossing there are a discrete number of possible outcomes; whereas with investments, the number of possible outcomes is continuous under certain distribution. In order to use the Kelly criterion in financial markets, the original model proposed on the Eq. (7) should be modified and adapted to represent continuous outcomes. This modification is proposed by Rotando and Thorp (1992). They propose a new model considering continuous outcome, but respecting the Kelly insight behind his original proposition, the definition of the optimal level of leverage in order to maximize the end of period wealth or the end of period return, $E[\log(Xn/X0)]$.

According to Rotando and Thorp (1992), trading financial securities can be considered a continuous game. Thus, in order to model the possible outcomes, they assume that financial returns is Normally distributed. However, the simple use of unaltered normal curve of probability distribution is inadequate because this distribution allows an infinite range of possible returns. Therefore, they modify the standard normal curve using a correction term for "chopping off the tails". It results in new parameters, h and α, which serves to maintain the mean and the standard deviation similar to standard normal curve.

Similar to binomial games, the objective in this model is to find out the level of leverage that maximizes function G(l), which in this case is:

$$GM_{rt}(l) = \int_A^B \log(1 + rl) dN(r) \tag{10}$$

$$GM_{rt}(l) = \int_A^B \log(1 + rl)[h + \frac{1}{\sqrt{2\pi\alpha^2}} e^{-(r-\mu)^2/2\alpha^2}] \tag{11}$$

where $A = \mu - 3\sigma$ and $B = \mu + 3\sigma$.

The demonstration of first-order and the second order conditions of the Eq. (11) is complicated, however numerical methods can be used to obtain the value of l that maximizes the function $GM_r t(l)$. For example, simulating the model proposed for 59 year period from 1926 to 1984, with the $\mu = 0.058$ and $\sigma = 0.216$; using numerical methods, (Rotando and Thorp 1992) found that the optimal level of leverage is $l^* = 1.7$.

3.3 Geometric Brownian Motion

Assuming that price follows a geometric Brownian motion, Peters(2009) proposes a different model to optimize the level of leverage. Let's suppose that the price process is:

$$p(t) = p_0 \exp{(\mu - \frac{\sigma^2}{2})t + \sigma * W(t)} \tag{12}$$

where the Wiener process W(t) is Gaussian-distributed.

Introducing the leverage, the estimated leveraged return and leveraged variance are respectively:

$$\mu_l = l\mu \tag{13}$$

$$\sigma_l^2 = l^2\sigma^2 \tag{14}$$

where l is the level of leverage.

The log-returned estimated of the levered investment to the next period is:

$$GM_p(l) = E[\log(p_{t+1})] - \log(p_t) = (\mu_l - \frac{\sigma^2}{2}) \tag{15}$$

Substituting Eq. (13) and Eq. (14) on Eq. (15):

$$GM_p(l) = (l\mu - \frac{l^2\sigma^2}{2}) \tag{16}$$

In order to obtain the optimal level of leverage, the Eq. (16) is differentiated with respect to l and the result is set to zero:

$$l^* = \frac{\mu}{\sigma^2} \tag{17}$$

where the optimal level of leverage l^* is obtained simply dividing the estimated return per its variance.

4 Numerical Methods

Similar to previous models, the model described on Eq. (17) ignores the presence of fat tails which is not ignored in the original approach of geometric mean proposed in the Eq. (5). It occurs because on the Taylor approximation used to get a model to calculate the geometric mean analytically, only the first two moments of the distribution are considered (Markowitz, 1991).

In general, every model presented above basically demonstrates the following characteristics about leverage: Firstly, geometric mean is concave in leverage, and consequently, there is an optimal level of leverage. Secondly, the optimal level of leverage is achieved analytically using those models.

The proposal of this paper is, instead of using analytical methods to obtain the optimal level of leverage, to return to the original model of geometric mean as demonstrated in the Eq. (5) and, to use a numerical method, in this case line-search, in order to obtain the optimal level of leverage.

In the line-search method, we randomly select two levels of leverage, l_1 and l_2, in which each point is on a different side of the maximum value.

Next, we select two new levels of leverage, l_3 and l_4. This will increase the geometric mean value. This procedure is performed up to get the specific level of leverage that no other level of leverage implies in higher geometric mean, which is considered the optimal level.

In order to use the numerical method, we transform the Eq. (5) and implement the term leverage on the geometric mean formula:

$$GM_{nm}(l) = [\prod_{t=1}^{n}(1 + lr_t)]^{\frac{1}{n}} - 1 \tag{18}$$

Where lr_t is the leveraged return at the time t. Sequentially, we use numerical method in order to obtain the value of l that maximizes the Eq. (18):

$$\underset{l}{\text{Max}}\, GM_{nm}(l) = [\prod_{t=1}^{n}(1 + lr_t)]^{\frac{1}{n}} - 1 \tag{19}$$

Numerical methods have three advantages. Firstly, it is not necessary to make any assumptions about the statistical characteristics of the data which helps us to avoid any further discussion about the realistic assumption of the distribution selected to represent the data. Secondly, the method is free of approximations or simplification, which also helps to make the methods the more realistic as possible. Finally, it is easy to be tested using real data.

5 Experiment

5.1 Description

In this section, we describe an experiment in which we test the ability of numerical methods to achieve the level of leverage that optimizes the geometric mean of the series of returns. As an empirical evidence, we use the series of daily return of the components of the Dow Jones index between 2003 and 2010. The data series contain 1024 days of 30 different stocks. We assume that our empirical data is sufficient to represent financial returns. Thus, due to the fact that financial series present common stylized fact, we agree that similar test could be made using different securities and periods of time, for example, the use of exchange rates between 1970 and 1985.

The experiment consists of a random selection of a window of a series of price for 10^4 rounds. In order to obtain the most representative data series during different periods of time, per round, we randomly select three parameters: the stock, the length of the series, and the period. For example, we randomly select the series of returns of GE between 01/02/2005 and 19/04/2007, as well as the series of returns of PFE between 23/08/2008 and 30/11/2010.

In every round, we calculated two different parameters: the optimal level of leverage and the optimal leveraged geometric mean using two different methods: the optimal level of leverage formula, as described in the on the Eq. (17), and the numerical methods as described on the Eq. (19). It generates four different data series content of 10^4 data each series: 1) the series of optimal level of leverage based on the geometric Brownian motion,

Item	$GM_{NM} - GM_{GBM}$
Confidence Interval (P≤95%)	[1.66e^{-}04;2.16e^{-}04]
Null Hypothesis	Rejected

Table 1: Comparision between geometric Brownian motion approach and numerical methods of the geometric mean using the optimal level of leverage .

2) the series of optimal level of leverage using numerical method, 3) the optimal leveraged geometric mean using numerical method, and 4) the optimal leveraged geometric mean based on the geometric Brownian motion.

5.2 Results

In order to compare the methods and its results, we test two different null hypotheses. Firstly, we test the null hypothesis that the optimal level of leverage using numerical method is similar to the optimal level of leverage based on geometric Brownian motion. Secondly, we test the null hypothesis that the leveraged geometric mean using numerical methods is similar to the leveraged geometric mean based on geometric Brownian motion.

From the Table 1, we can see that the confidence interval of the difference between the series of geometric means calculated using the numerical methods and the series of geometric means using the geometric Brownian motion is positive. It demonstrates that numerical methods yield to a higher value of geometric mean than the geometric Brownian motion with a probability of 97.5%.

Furthermore, from the Table 2, similar to the Table 1, we can see that the confidence interval of the difference between the series of the level of leverage

Item	$L_{NM} - L_{GBM}$
Confidence Interval (P≤95%)	[0.1934; 0.2428]
Null Hypothesis	Rejected

Table 2: Comparision between geometric Brownian motion approach and numerical methods of the optimal level of leverage.

that were calculated using numerical methods and the series using geometric Brownian motion is positive. It demonstrates that the numerical methods yield to higher value of level of leverage than the geometric Brownian with a probability of 97.5%.

6 Conclusion

Advances in computer software allow the use of numerical methods directly on the formula, thus eliminating restrictions that used to be necessary in order to optimize it. In this paper, we propose a numerical method in order to find the level of leverage which maximizes the geometric mean formula. Furthermore, we demonstrate that numerical methods are more appropriate than analytical methods to calculate the optimal level of leverage because it leads to a superior geometric mean.

For future work, there are two different fields which our research can draw upon. The first field is the back-testing of the ability of the optimal level of leverage to improve the investment performance; and the second field is the use of numerical method to determine the ideal level of leverage of a portfolio of investments instead of an isolated security.

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