GENERAL MULTIVARIATE DEPENDENCE USING ASSOCIATED COPULAS

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Abstract. This paper studies the general multivariate dependence of a random vector using associated copulas. We extend definitions and results of positive dependence to the general dependence case. This includes associated tail dependence functions and associated tail dependence coefficients. We derive the relationships among associated copulas and study the associated copulas of the perfect dependence cases and elliptically contoured distributions. We present the expression for the associated tail dependence function of the multivariate Student-t copula, which accounts for all types of tail dependence.

1. Introduction

A great deal of literature has been written on the analysis of the dependence structure between random variables. There is an increasing interest in the understanding of the dependencies between extreme values in what is known as tail dependence. However, the analysis of multivariate tail dependence has been exclusively focused on the positive case, leaving a void in the analysis of dependence structure. In this paper we address this issue by defining the concepts needed to measure a general type of tail dependence in the multivariate case. We use a copula approach and base our study on the associated copulas (see Joe (1997), p. 15).

The dependence structure of time series has been studied for a long time, traditionally through the use of the Pearson's correlation coefficient. More recently, copula based measures

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Date: April 20th, 2012.

This paper is based on results from the first chapters of my doctorate thesis supervised by Dr. Wing Lon Ng. I would like to thank Mexico's CONACYT, for the funding during my studies. I would also like to thank my examiners Dr. Aristidis Nikoloulopoulos and Dr. Nick Constantinou for the corrections, suggestions and comments that have made this work possible.

such as the Spearman's ρ and Kendall's τ have also been used to assess concordance. Due to drawbacks of these measures when it comes to tail dependence, new methodologies have been developed. In particular the use of the tail dependence coefficient (TDC) and the tail dependence function has proven to be the way forward (see Nelsen (1999), Joe (1997) and McNeil et al. (2005), Chapter 5). In the multivariate case positive tail dependence has been analysed by the use of the copula C and the survival copula $\widehat{C}$ and the use of the L (lower) and U (upper) multivariate tail dependence functions introduced by Nikoloulopoulos et al. (2009). However it is sometimes important to analyse non-positive tail dependence.

In quantitative finance, for example, the performance of portfolios of multiple stocks is highly dependent on the stocks being able to hedge extreme movements. If, for instance, during a crisis the price of ten of the stocks of a portfolio drops it is desirable to have stocks whose prices rise. In this case the presence of multivariate non-positive tail dependence is desirable. Knowing the tail dependence structure among several stocks can help determine the weights of the different stocks of a portfolio.

In order to address non-positive dependence, we introduce the concept of general dependence $\mathbf{D}$, along with other concepts such as $\mathbf{D}$-probability functions. It is through these functions that copula theory can be extended to account for non-positive dependence. We prove that the copulas that link $\mathbf{D}$-probability functions and its marginals are the associated copulas. We use these copula to define general tail dependence. All the results presented regarding general dependence are also a contribution of this work. This includes the relationship among associated copulas, the monotonic copulas, the associated elliptical copulas and the associated tail dependence functions of the Student-t copula model.

The reminder of this work is divided in three sections: In the second section we present the corresponding concepts of general dependence. This includes probability functions, copulas, tail dependence functions and TDCs, mathematical proofs are provided on main results. We present the equalities connecting all associated copulas and results regarding monotone functions. In the third section we study the associated copulas of the perfect dependence cases and the elliptically contoured distributions introduced by Kelker (1970). We present the associated tail dependence function of the Student-t copula, which is positive for all types of dependence. In the fourth section we conclude.

2. Associated Copulas, Tail Dependence Functions and the Tail Dependence Coefficients

In this section we analyse the dependence structure among random variables using copulas. Given a random vector $\mathbf{X} = (X_1, ..., X_d)$, we use the corresponding copula C and its associated copulas to analyse its dependence structure. For this we introduce a general type of dependence $\mathbf{D}$, that corresponds to the lower and upper movements of the different variables.

To analyse different dependencies, we introduce the $\mathbf{D}$-probability function and present a version of Sklar's theorem that states that an associated copula is the copula that link this function and its marginals. We present a formula to link all associated copulas and two results on monotone functions and associated copulas. We then present the associated tail dependence function and the associated tail dependence coefficient (TDC) for the type of dependence $\mathbf{D}$ (see Joe (1993), Joe (1997) and Nikoloulopoulos et al. (2009)). With the concepts studied in this section, it is possible to analyse the whole dependence structure among random variables, this includes non-positive dependence.

2.1. Copulas and Dependence D.

The concept of copula was first introduced by Sklar (1959), and is now a cornerstone topic in multivariate dependence analysis (see Joe (1997), Nelsen (1999) and McNeil et al. (2005), Chapter 5). We now present the concepts of copula, general dependence and associated copulas that are fundamental for the rest of this work.

Definition 1. *A multivariate copula* $C(u_1, ..., u_d)$ *is a distribution function on the* d*-dimensional-square* $[0,1]^d$ *with standard uniform marginal distributions.*

If C is the distribution function of $\mathbf{U} = (U_1, ..., U_d)$, we denote as $\widehat{C}$ the distribution function of $(1 - U_1, ..., 1 - U_d)$. In the multivariate case, C is used to link multivariate distribution functions with their corresponding marginal distributions, accordingly we refer to C as the distributional copula. On the other hand, $\widehat{C}$ is used to link multivariate survival functions with their corresponding marginal survival functions, this copula is known as the survival copula.[1] The survival copula $\widehat{C}$ must not be confused with the survival function $\overline{C}(u_1, ..., u_d) = \widehat{C}(1 - u_1, ..., 1 - u_d)$.

Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with joint distribution function

$$F(x_1, ..., x_d) = P(X_1 \leq x_1, ..., X_d \leq x_d),$$

survival distribution function

$$\overline{F}(x_1, ..., x_d) = P(X_1 > x_1, ..., X_d > x_d),$$

marginals $F_i(x_i) = P(X_i \leq x_i)$ and marginal survival functions $\overline{F_i}(x_i) = P(X_i > x_i)$ for all $i \in \{1, ...d\}$. Sklar's theorem guarantees the existence and uniqueness of copulas C and $\widehat{C}$ such that

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)),$$

which is equivalent to

$$C(u_1, ..., u_d) = F(F_1^{\leftarrow}(u_1), ..., F_d^{\leftarrow}(u_d)) \tag{2.1}$$

(see Joe (1997)). Similarly, Sklar's theorem for survival functions implies

$$\overline{F}(x_1, ..., x_d) = \widehat{C}(\overline{F}_1(x_1), ..., \overline{F}_d(x_d)),$$

which is equivalent to

$$\widehat{C}(u_1, ..., u_d) = \overline{F}(\overline{F_1}^{\leftarrow}(u_1), ..., \overline{F_d}^{\leftarrow}(u_d)) \tag{2.2}$$

(Georges et al. (2001)).

In the next section we generalise these equations using the concept of a type of dependence in the general case, which we now define.

Definition 2. *In d dimensions, we call the vector $\mathbf{D} = (D_1, ..., D_d)$ a type of dependence if each D_i is a boolean variable, whose value is either L (lower) or U (upper) for $i \in \{1, ...d\}$. We denote by Δ the set of all 2^d types of dependence.*

[1]We use the term distributional for C, to distinguish it from the other associated copulas. The notation for the survival copula corresponds to the one used in the seminal work of Joe (1997).

Two well known types of dependence are the lower dependence, which corresponds to the case $D_i = L$ for $i \in \{1, ..., d\}$, and upper dependence, which corresponds to the case $D_i = U$ for $i \in \{1, ..., d\}$ (see e.g. Joe (1997) and Nelsen (1999)). These are examples of positive dependence. In the bivariate case the dependencies $\mathbf{D} = (L, U)$ and $\mathbf{D} = (U, L)$ correspond to negative dependence, which is often present in financial time series, see Zhang (2007), Embrechts et al. (2009) and Joe (2011).

Using the concept of dependence, we now present the associated copulas introduced by Joe (1997), Chapter 1, p.15.

Definition 3. *Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with corresponding copula C, which is the distribution function of the vector $(U_1, ..., U_d)$ with uniform marginals. Let Δ denote the set of all types of dependences of Definition 2. For $\mathbf{D} = (D_1, ..., D_d) \in \Delta$, let $\mathbf{V_D} = (V_{D_1,1}, ..., V_{D_d,d})$ with*

$$V_{D_i,i} = \begin{cases} U_i & \text{if } D_i = L \\ 1 - U_i & \text{if } D_i = U \end{cases} .$$

Note that $\mathbf{V_D}$ also has uniform marginals. We call the distribution function of $\mathbf{V_D}$, which is also a copula, the associated $\mathbf{D}$-copula and denote it $C_{\mathbf{D}}$. We denote $\mathcal{A}_{\mathbf{X}} = \{C_{\mathbf{D}} | \mathbf{D} \in \Delta\}$, the set of 2^d associated copulas of the random vector $\mathbf{X}$.

Note that the distributional and the survival copula are $C = C_{(L,...,L)}$ and $\widehat{C} = C_{(U,...,U)}$ respectively.

2.1.1. *The D-Probability Function and its Associated D-Copula.*

The distributional copula C and the survival copula $\widehat{C}$ are used to explain the lower and upper dependence structure of a random vector respectively. We use the associated $\mathbf{D}$-copula to explain the $\mathbf{D}$-dependence structure of a random vector. For this, we first present the concept of $\mathbf{D}$-probability function and then prove that the associated copula $C_{\mathbf{D}}$ links this function with its marginals.

Definition 4. *Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with marginal distributions F_i for $i \in \{1, ...d\}$ and $\mathbf{D} = (D_1, ..., D_d)$ a type of dependence according to Definition 2. Define the* *event $\mathcal{B}_i(x_i)$ in the following way*

$$\mathcal{B}_i(x_i) = \begin{cases} \{X_i \leq x_i\} \text{ if } D_i = L \\ \{X_i > x_i\} \text{ if } D_i = U \end{cases}.$$

Then the corresponding **D***-probability function is*

$$F_{\mathbf{D}}(x_1, ..., x_d) = P\left(\bigcap_{i=1}^{d} \mathcal{B}_i(x_i)\right).$$

We refer to

$$F_{D_i,i} = \begin{cases} F_i \text{ if } D_i = L \\ \overline{F_i} \text{ if } D_i = U \end{cases},$$

for $i \in \{1, ...d\}$ as the marginal functions of $F_{\mathbf{D}}$ (Note that the marginals are either univariate distribution or survival functions).

The copula we consider to analyse the **D**-dependence is $C_{\mathbf{D}}$ that link the functions in Definition 4 with their corresponding marginals. Given that by applying decreasing transformations to a part of the data we can account for negative dependence the copulas of the **D**-probability functions correspond to the associated copulas of Definition 3. The following theorem presents the associated copula $C_{\mathbf{D}}$ in terms of the $F_{\mathbf{D}}$ and its marginals. We restrict the proof to the continuous case (see McNeil et al. (2005), p.186, Joe (1997) or Nelsen (1999)).

Theorem 1. ***Sklar's theorem for* D*-probability functions and associated copulas***

Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector, $\mathbf{D} = (D_1, ..., D_d)$ a type of dependence, $F_{\mathbf{D}}$ its **D***-probability function and $F_{D_i,i}$ for $i \in \{1, ...d\}$ the marginal functions of $F_{\mathbf{D}}$ as in Definition 4. Let the marginal functions of $F_{\mathbf{D}}$ be continuous, then the associated copula $C_{\mathbf{D}} : [0,1]^d \to [0,1]$, satisfies, for all $x_1, ..., x_2$ in $[-\infty, \infty]$,*

$$F_{\mathbf{D}}(x_1, ..., x_d) = C_{\mathbf{D}}(F_{D_1,1}(x_1), ..., F_{D_d,d}(x_d)), \tag{2.3}$$

which is equivalent to

$$C_{\mathbf{D}}(u_1, ..., u_d) = F_{\mathbf{D}}(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d)). \tag{2.4}$$

Conversely, let $\mathbf{D} = (D_1, ..., D_d)$ be a type of dependence and $F_{D_i,i}$ a univariate distribution function, if $D_i = L$, or a survival function, if $D_i = U$, $i \in \{1, ...d\}$,

(a) *if* $C_{\mathbf{D}}$ *is a copula, then* $F_{\mathbf{D}}$ *in (2.3) defines a* $\mathbf{D}$*-probability function with marginals* $F_{D_i,i}$, $i \in \{1, ...d\}$.

(b) *if* $F_{\mathbf{D}}$ *is any* $\mathbf{D}$*-probability function, then* $C_{\mathbf{D}}$ *in (2.4) is a copula.*

Proof. The proof of this theorem is analogous to the proof of Sklar's theorem in the continuous case. In the continuous case for any distribution function F_i, we have that the events $\{X_i \leq x_i\} \overset{P}{\sim} \{F_i(X_i) \leq F_i(x_i)\}$ and $\{X_i > x_i\} \overset{P}{\sim} \{\overline{F}_i(X_i) \leq \overline{F}_i(x_i)\}$. This implies

$$P(\mathcal{B}_i(x_i)) = P(F_{D_i,i}(X_i) \leq F_{D_i,i}(x_i)), \tag{2.5}$$

for $i \in \{1, ..., d\}$.

Considering equation (2.5) and Definition 4, we have that for any $x_1, ..., x_d$ in $[-\infty, \infty]$

$$F_D(x_1, ..., x_d) = P(F_{D_1,1}(X_1) \leq F_{D_1,1}(x_1), ..., F_{D_d,d}(X_d) \leq F_{D_d,d}(x_d)). \tag{2.6}$$

Using the continuity of F_i we have that $F_i(X_i)$ is uniformly distributed (see McNeil et al. (2005), Proposition (5.2 (2)), p.185). Hence, if we define $\mathbf{U} = (F_1(X_1), ..., F_d(X_d))$, its distribution function is a copula C. Note that in this case $\mathbf{V_D}$, defined as in Definition 3, is equal to $(F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d))$. It follows that the distribution function of $(F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d))$ is the associated copula $C_{\mathbf{D}}$, in which case equation (2.5) implies

$$C_{\mathbf{D}}(F_{D_1,1}(x_1), ..., F_{D_d,d}(x_d)) = P(F_{D_1,1}(X_1) \leq F_{D_1,1}(x_1), ..., F_{D_d,d}(X_d) \leq F_{D_d,d}(x_d)),$$

and equation (2.3) follows.

If we evaluate $F_{\mathbf{D}}$ in $(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d))$, we get

$$C_{\mathbf{D}}(u_1, ..., u_d) = F_D(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d)).$$

This follows from the fact that one of the properties of the generalized inverse is that, when T is continuous, $T \circ T^{\leftarrow}(x) = x$ (see McNeil et al. (2005), Proposition (A.3)). This equation explicitly represents $C_{\mathbf{D}}$ in terms of $F_{\mathbf{D}}$ and its marginals implying its uniqueness.

For the converse statement of the theorem, we have

(a) Let $\mathbf{U} = (U_1, ..., U_d)$ be the random vector with distribution function C. We now define

$$\begin{aligned} \mathbf{X} &= (X_1, ..., X_d) \\ &= ((F^{\leftarrow}_{D_1,1}(U_1), ..., F^{\leftarrow}_{D_d,d}(U_d))), \end{aligned}$$

and

$$\mathcal{B}_i(x_i) = \begin{cases} \{X_i \le x_i\} \text{ if } D_i = L \\ \{X_i > x_i\} \text{ if } D_i = U \end{cases},$$

for $i \in \{1, ...d\}$. Considering that $F(x) \le y \iff x \le F^{\leftarrow}(y)$, we have $\overline{F}^{\leftarrow}(x) \le y \iff x \ge \overline{F}(y)$. Using these properties, we have

$$\{U_i \le F_{D_i,i}(x_i)\} \overset{P}{\sim} \mathcal{B}_i(x_i),$$

for $i \in \{1, ...d\}$. Using this, the $\mathbf{D}$-probability function of X is

$$P\left(\bigcap_{i=1}^{d} \mathcal{B}_i(x_i)\right) = C(F_{D_1,1}(x_1), ..., F_{D_d,d}(x_d)).$$

This implies that $F_{\mathbf{D}}$ defined by (2.3) is the $\mathbf{D}$-probability function of X with marginals

$$P(\mathcal{B}_i(x_i)) = P(U_i \le F_{D_i,i}(x_i)) = F_{D_i,i}(x_i),$$

for $i \in \{1, ...d\}$.

(b) Similarly, let $(X_1, ..., X_d)$ be the random vector with $\mathbf{D}$-probability function $F_{\mathbf{D}}$. Define

$$\begin{aligned} \mathbf{U} &= (U_1, ..., U_d) \\ &= (F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d)) \end{aligned}$$

(note that the vector is uniformly distributed). Again, using the properties of the generalised inverse, we have

$$\{U_i \le u_i\} \overset{P}{\sim} \mathcal{B}_i(F^{\leftarrow}_{D_i,i}(u_i)),$$

for $i \in \{1, ...d\}$. Hence the distribution function of $\mathbf{U}$ is $F_{\mathbf{D}}(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d))$, which implies that the function is a copula.

□

For this theorem we refer to generalized inverses rather than inverse functions, as the first are more general. However throughout this work, whenever we are not proving a general property, we assume the distribution functions have inverse functions. For the properties of the generalized inverse function used in this proof, see McNeil et al. (2005), Proposition (A.3).

Note that this theorem implies that in the continuous case $C_{\mathbf{D}}$ is the $\mathbf{D}$-probability function of $(F_{D_1,1}(X_1),...,F_{D_d,d}(X_d))$ characterised in (2.3). This theorem implies the importance of the associated copulas to analyse dependencies. It also implies the Fréchet bounds for the $\mathbf{D}$-probability functions of Definition 4. The bounds can also be obtained similarly to Joe (1997), Theorems 3.1 and 3.5 (p. 58 and 59):

$$\begin{aligned}\max\{0,F_{D_1,1}(x_1)+...+F_{D_d,d}(x_d)-(d-1)\} \;\leq\;& F_{\mathbf{D}}(x_1,...,x_d)\leq \\ & \min\{F_{D_1,1}(x_1),...,F_{D_d,d}(x_d)\}\,.\end{aligned} \tag{2.7}$$

2.1.2. *Properties of the Associated D-Copulas.*

In the bivariate case, Joe (1997), Chapter 1, p. 15, and Nelsen (1999), Chapter 2, p. 26, presented the expressions to link the associated copulas with the distributional copula C. In the multivariate case Joe (2011), Equation 8.1, p. 200, and Georges et al. (2001), Theorem 3, p. 7, presented the expression between the distributional and the survival copula and Embrechts et al. (2001), Theorem 2.7, p. 6, proved that is possible to express the associated copulas in terms of the distributional copula C. We now present a general equation for the relationship between any two associated copulas $C_{\mathbf{D}^*}$ and $C_{\mathbf{D}^+}$ in the multivariate case. The equation is based on all the subsets of the indices where the $\mathbf{D}^*$ and $\mathbf{D}^+$ are different. After this, we prove that the associated copulas are invariant under strictly increasing transformations and characterise the copula after strictly monotone transformations.

Proposition 1. *Let* $\mathbf{X}=(X_1,...,X_d)$ *be a random vector with associated copulas* $\mathcal{A}_{\mathbf{X}}$ *and* $\mathbf{D}^*=(D_1^*,...,D_d^*)$ *and* $\mathbf{D}^+=(D_1^+,...,D_d^+)$ *any two types of dependence. Consider the following sets and notations:* $I=\{1,...,d\}$; $I_1=\{i\in I|D_i^*=D_i^+\}$ *and* $I_2=\{i\in I|D_i^*\neq D_i^+\}$; $d_1=|I_1|$ *and* $d_2=|I_2|$; $S_j=\{\text{the subsets of size } j \text{ of } I_2\}$ *and* $S_{j,k}=\{\text{The } k\text{-th element of } S_j\}$ *for* $j\in\{1,...,d_2\}$ *and* $k\in\{1,...,\binom{d_2}{j}\}$. *We define* $S_0=\emptyset$ *and* $S_{0,1}=\emptyset$; *for each* $S_{j,k}$ *define* $\mathbf{W}_{j,k}=(W_{j,k.1},...,W_{j,k,d})$ *with*

$$W_{j,k,i}=\begin{cases} u_i & \text{if } i\in I_1\\ 1-u_i & \text{if } i\in S_{j,k}\\ 1 & \text{if } i\notin I_1\cup S_{j,k}\end{cases},$$

for $i \in \{1, ...d\}$, $j \in \{0, ..., d_2\}$ *and* $k \in \{1, ..., \binom{d_2}{j}\}$.

Then the associated $\mathbf{D}^*$*-copula* $C_{\mathbf{D}^*}$ *is expressed in terms of the* $\mathbf{D}^+$*-copula* $C_{\mathbf{D}^+}$ *according to the following equation*

$$C_{\mathbf{D}^*}(u_1, ..., u_d) = \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^+}(\mathbf{W}_{j,k}). \tag{2.8}$$

Note that in the cases when at least a 1 *appears in* $\mathbf{W}_{j,k}$, $C_{\mathbf{D}^+}(\mathbf{W}_{j,k})$ *becomes a marginal copula of* $C_{\mathbf{D}^+}$.

Proof. Throughout this proof, it must be borne in mind that $C_{\mathbf{D}^*}$ is the distribution function of the random vector $\mathbf{V}_{\mathbf{D}^*}$ and $C_{\mathbf{D}^+}$ of $\mathbf{V}_{\mathbf{D}^+}$, defined according to Definition 3. Note that, for $i \in I_2$, $V_{D_i^*,i} = 1 - V_{D_i^+ i}$ and they are equal otherwise.

In the case $d_2 = 0$, we have $\mathbf{D}^* = \mathbf{D}^+$, $j \in \{0\}$ and $k \in \{1\}$[2], hence (2.8) holds. We prove (2.8) by induction on d, the dimension; it can also be proven by induction on d_2, the number of elements in which $D_i^* \neq D_i^+$. Note that in dimension $d = 1$, a copula becomes the identity function. If $D_1^* \neq D_1^+$, the expression becomes $u_1 = 1-(1-u_1)$; the case $D_1^* = D_1^+$ has already been covered in $d_2 = 0$, and expression (2.8) holds.

Now, suppose we are in dimension d, we prove the formula works provided it works in dimension $d-1$. We obtain an expression for $C_{\mathbf{D}^*}(u_1, ..., u_d)$ using the induction hypothesis. Consider the dependencies, on the $(d-1)$-dimension, $\mathbf{F}^* = (D_1^*, ..., D_{d-1}^*)$ and $\mathbf{F}^+ = (D_1^+, ..., D_{d-1}^+)$. We use an apostrophe on the sets and notations of $\mathbf{F}^*$ and $\mathbf{F}^+$ to differentiate them from those of $\mathbf{D}^*$ and $\mathbf{D}^+$. It follows that $d' = d-1$ and $I' = I - \{d\}$. By the induction hypothesis, equation (2.8) holds to express $C_{\mathbf{F}^*}$ in terms of $C_{\mathbf{F}^+}$. In terms of probabilities this is equivalent to

$$\begin{aligned} &P(V_{D_i^*,1} \leq u_1, ..., V_{D_{d-1}^*,d-1} \leq u_{d-1}) \\ &= \sum_{j=0}^{d_2-1} (-1)^j \sum_{k=1}^{\binom{d_2-1}{j}} P(V_{D_1^+,1} \leq W'_{j,k,1}, ..., V_{D_{d-1}^+,d-1} \leq W'_{j,k,d-1}), \end{aligned} \tag{2.9}$$

Now there are two cases to consider depending on whether D_d^* is equal to D_d^+ or not.

Case 1. $D_d^* = D_d^+$.

[2]Note that we are using the convention $0! = 1$

In this case, it follows that, $I_1' = I_1 - \{d\}$, $I_2' = I_2$, $d_2' = d_2$ *and* $V_{D_d^*,d} = V_{D_d^+,d}$. *If we intersect the events in equation (2.9) with the event* $\{V_{D_d^* d} \le u_d\}$ *we get*

$$\begin{aligned} &P(V_{D_i^*,1} \le u_1, ..., V_{D_{d-1}^*,d-1} \le u_{d-1}, V_{D_d^* d} \le u_d) \\ &= \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} P(V_{D_1^+,1} \le W'_{j,k,1}, ..., V_{D_{d-1}^+,d-1} \le W'_{j,k,d-1}, V_{D_d^+,d} \le u_d). \end{aligned} \tag{2.10}$$

Because $I_2' = I_2$, *in this case, for* $j \in \{1, ..., d_2\}$ *and* $k \in \{1, ..., \binom{d_2}{j}\}$, *the events* $S'_{j,k}$ *are equal to* $S_{j,k}$. *Considering this, and* $I_1' = I_1 - \{d\}$, *we have*

$$(\mathbf{W'}_{j,k}, u_d)_i = W_{j,k,i}$$

for $i \in \{1, ..., d\}$, *so* $(\mathbf{W'}_{j,k}, u_d) = \mathbf{W}_{j,k}$ *for* $j \in \{1, ..., d_2\}$ *and* $k \in \{1, ..., \binom{d_2}{j}\}$. *Equation (2.10) then implies:*

$$C_{D^*}(u_1, ..., u_d) = \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{D^+}(\mathbf{W}_{j,k}).$$

Case 2. $D_d^* \ne D_d^+$

In this case, it holds that, $I_1' = I_1$, $I_2' = I_2 - \{d\}$, $d_2' = d_2 - 1$. *We want to obtain an expression for* $C_{\mathbf{D}^*}(u_1, ..., u_d) = P(V_{D_i^*,1} \le u_1, ..., V_{D_d^*,d} \le u_d)$, *using the induction hypothesis. Considering that, in general,* $P(A) = P(A \cap B) + P(A \cap B^c)$ *we have that*

$$\begin{aligned} P(V_{D_i^*,1} \le u_1, ..., V_{D_{d-1}^*,d-1} \le u_{d-1}) &= P(V_{D_i^*,1} \le u_1, ..., V_{D_{d-1}^*,d-1} \le u_{d-1}, V_{D_d^*,d} \le u_d) \\ &+ P(V_{D_i^*,1} \le u_1, ..., V_{D_{d-1}^*,d-1} \le u_{d-1}, V_{D_d^*,d} \ge u_d), \end{aligned}$$

which implies

$$C_{\mathbf{D}^*}(u_1, ..., u_d) = P(V_{D_1^*,1} \le u_1, ..., V_{d-1}^* \le u_{d-1}) - P(V_{D_1^*,1} \le u_1, ..., V_{d-1}^* \le u_{d-1}, V_{D_d^*,d} \ge u_d). \tag{2.11}$$

Note that, in this case $V_{D_d^*,d} = 1 - V_{D_d^+,d}$. *This implies that the event* $\{V_{D_d^*,d} \ge u_d\}$ *is equivalent to* $\{V_{D_d^+,d} \le 1 - u_d\}$. *If we intersect the events involved in equation (2.9) with the event* $\{V_{D_d^*,d} \ge u_d\}$ *we get*

$$\begin{aligned} &P(V_{D_1^*,1} \le u_1, ..., V_{D_{d-1}^*,d-1} \le u_{d-1}, V_{D_d^*,d} \ge u_d) = \\ &\sum_{j=0}^{d_2-1} (-1)^j \sum_{k=1}^{\binom{d_2-1}{j}} P(V_{D_1^+,1} \le W'_{j,k,1}, ..., V_{D_{d-1}^+,d-1} \le W'_{j,k,d-1}, V_{D_d^+,d} \le 1 - u_d). \end{aligned} \tag{2.12}$$

Combining equations (2.9), (2.11) and (2.12), we obtain

$$C_{\mathbf{D}^*}(u_1,...,u_d) = \sum_{j=0}^{d_2-1}(-1)^j \sum_{k=1}^{\binom{d_2-1}{j}} C_{\mathbf{D}^+}(\mathbf{W}'_{j,k},1) - \sum_{j=0}^{d_2-1}(-1)^j \sum_{k=1}^{\binom{d_2-1}{j}} C_{\mathbf{D}^+}(\mathbf{W}'_{j,k},1-u_d). \tag{2.13}$$

Note that, in this case, the sets I_2 and I_2' satisfy $I_2 = I_2' \cup \{d\}$.

The rest of the proof is based on the fact that for $j \in \{1,...,d-1\}$ the elements of size j of I_2 are the elements of size j of I_2' plus the elements of size $j-1$ of I_2' attaching them $\{d\}$. Considering our notation, this means

$$S_j = S_j' \cup S''_{j-1}, \tag{2.14}$$

with $S''_{j-1} = \left\{ S''_{j-1,k} = S'_{j-1,k} \cup \{d\} \middle| \, k \in \{1,..., \binom{d_2-1}{j-1}\} \right\}$ for $j \in \{1,...,d-1\}$. Further to this, by definition of $\mathbf{W}_{j,k}$ we have the following two equalities

$$(\mathbf{W}'_{j,k},1)_i = \begin{cases} u_i & \text{if } i \in I_1 \\ 1-u_i & \text{if } i \in S'_{j,k} \\ 1 & \text{if } i \notin I_1 \cup S'_{j,k} \end{cases} \tag{2.15}$$

and

$$(\mathbf{W}'_{j-1,k},1-u_d)_i = \begin{cases} u_i & \text{if } i \in I_1 \\ 1-u_i & \text{if } i \in S''_{j-1,k} \\ 1 & \text{if } i \notin I_1 \cup S''_{j-1,k} \end{cases}, \tag{2.16}$$

for $i \in \{1,...d\}$, $j \in \{1,...,d-1\}$ and $k \in \{1,..., \binom{d_2-1}{j-1}\}$.

Given that

$$W_{j,k,i} = \begin{cases} u_i & \text{if } i \in I_1 \\ 1-u_i & \text{if } i \in S_{j-1,k} \\ 1 & \text{if } i \notin I_1 \cup S_{j,k} \end{cases},$$

equations (2.14) to (2.16) imply that, for a fixed j, if we sum $C_{\mathbf{D}^+}$ evaluated in all of the $(\mathbf{W}'_{j,k},1)$ and $(\mathbf{W}'_{j,k},1-u_d)$ for different k, we get the sum of $C_{\mathbf{D}^+}$ evaluated on $\mathbf{W}_{j,k}$ for different k, that is:

$$\sum_{k=1}^{\binom{d_2-1}{j}} C_{\mathbf{D}^+}(\mathbf{W}'_{j,k},1) + \sum_{k=1}^{\binom{d_2-1}{j-1}} C_{\mathbf{D}^+}(\mathbf{W}'_{j-1,k},1-u_d) = \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^+}(\mathbf{W}_{j,k}), \tag{2.17}$$

for $j \in \{1, ..., d-1\}$. Also, note that

$$(\mathbf{W}'_{0,1}, 1)_i = W_{0,1,i}, \tag{2.18}$$

and

$$(\mathbf{W}'_{d-1,1}, 1-u_d)_i = W_{d,1,i}, \tag{2.19}$$

for $i \in \{1, ...d\}$; the result is implied by equation (2.13) and equations (2.17) to (2.19). □

Note that this expression is mainly dependent on the subsets of I_2, the elements in which $D_i^* \neq D_i^+$. Because of this, the expression is reflexible, meaning that it yields the same formula to express $C_{\mathbf{D}^+}$ in terms of $C_{\mathbf{D}^*}$. The formula has 2^{d_2} elements (one for every $S_{j,k}$). In particular, equation (2.8) can be used to express any associated copula in terms of the distributional copula C. This is useful considering that the expression found in literature for copula models is the one for the distributional copula.

Corollary 1. *Let $X = (X_1, ..., X_d)$ be a random vector with copula C and $\mathbf{D} = (D_1, ..., D_d)$ a type of dependence. Consider the same notations of proposition (1) with $I_1 = \{i \in I | D_i = L\}$ and $I_2 = \{i \in I | D_i = U\}$. Then the associated $\mathbf{D}$-copula $C_{\mathbf{D}}$ is expressed in terms of C according to*

$$C_{\mathbf{D}}(u_1, ..., u_d) = \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C(\mathbf{W}_{j,k}).$$

In order to analyse the symmetry and exchangeability of copula models, we use the following definitions.

Definition 5. *Let $\mathbf{D} = (D_1, ..., D_d)$ be a type of dependence, the complement dependence is defined as $\mathbf{D}^{\complement} = (D_1^{\complement}, ..., D_d^{\complement})$, with*

$$D_i^{\complement} = \begin{cases} U \text{ if } D_i = L \\ L \text{ if } D_i = U \end{cases},$$

for $i \in \{1, ..., d\}$. We say that the random vector X, with associated copulas $\mathcal{A}_{\mathbf{X}}$, is complement (reflection or radial) symmetric, if there exists $\mathbf{D}^* \in \Delta$, such that $C_{\mathbf{D}^*} = C_{\mathbf{D}^{*\complement}}$.

Definition 6. *A random vector $X = (X_1, ..., X_d)$ is said to be exchangeable if, for every permutation PR of $\{1, ..., d\}$, $PR(i) = p_i$, it holds that $(X_1, ..., X_d) \stackrel{d}{=} (X_{p_1}, ..., X_{p_d})$. A copula C is said to be exchangeable if it is the distribution function of an exchangeable vector, in which case, the copula satisfies $C(u_1, ..., u_d) = C(u_{p_1}, ..., u_{p_d})$ for every permutation.*

In the following proposition we obtain equivalences for the exchangeability and equalities regarding associated copulas. According to proposition (1), the relationship between two associated copulas $C_{\mathbf{D}^*}$ and $C_{\mathbf{D}^+}$ is determined by the elements in which $\mathbf{D}^*$ and $\mathbf{D}^+$ are different. Such elements are denoted as I_2, given that we deal with several types of dependence, we denote this set as $I_2(\mathbf{D}^*, \mathbf{D}^+)$ to indicate the dependencies to which it refers. We do the same for $I_1(\mathbf{D}^*, \mathbf{D}^+)$, the elements in which the dependencies are equal.

Proposition 2. *Let $\mathbf{X}$ be a vector with corresponding associated copulas $\mathcal{A}_{\mathbf{X}}$, and let $\mathbf{D}^*$, $\mathbf{D}^+$, $\mathbf{D}^\circ$ and $\mathbf{D}^\times$ be types of dependencies. Then the following equivalences hold:*

(i) *If $C_{\mathbf{D}^*} \equiv C_{\mathbf{D}^+}$ and $I_2(\mathbf{D}^*, \mathbf{D}^+) = I_2(\mathbf{D}^\times, \mathbf{D}^\circ)$ then $C_{\mathbf{D}^\times} \equiv C_{\mathbf{D}^\circ}$.*
In particular $C_{\mathbf{D}^*} \equiv C_{\mathbf{D}^{*\complement}}$, for some $\mathbf{D}^*$, implies $C_{\mathbf{D}} \equiv C_{\mathbf{D}^\complement}$ for all $\mathbf{D} \in \Delta$.

(ii) *If $C_{\mathbf{D}^\circ}$ is exchangeable, then the following hold:*

(a) *$C_{\mathbf{D}^*}$ is exchangeable over the elements of $I_1(\mathbf{D}^*, \mathbf{D}^\circ)$ and over the elements of $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$.*
In particular, if $C_{\mathbf{D}^\circ}$ is exchangeable, then $C_{\mathbf{D}^{\circ\complement}}$ is exchangeable.

(b) *If $|I_2(\mathbf{D}^*, \mathbf{D}^\circ)| = |I_2(\mathbf{D}^+, \mathbf{D}^\circ)|$, let PR be any permutation of $\{1, ..., d\}$ that assigns to each element of $I_2(\mathbf{D}^+, \mathbf{D}^\circ)$, an element of $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$. Denote $i' = PR(i)$, then*

$$C_{\mathbf{D}^*}(u_1, ..., u_d) = C_{\mathbf{D}^+}(u_{1'}, ..., u_{d'}).$$

(c) *If d is even and there exists $C_{\mathbf{D}^*}$ exchangeable, such that $|I_2(\mathbf{D}^*, \mathbf{D}^\circ)| = \frac{d}{2}$ then $C_{\mathbf{D}} \equiv C_{\mathbf{D}^\complement}$ for all $\mathbf{D} \in \Delta$.*

Proof. (i) This follows from the fact $I_2(\mathbf{D}^*, \mathbf{D}^+) = I_2(\mathbf{D}^\times, \mathbf{D}^\circ) \implies I_2(\mathbf{D}^\times, \mathbf{D}^*) = I_2(\mathbf{D}^\circ, \mathbf{D}^+)$, which is easily verified considering the different cases. From proposition (1), we have that the vectors $\mathbf{W}_{j,k}$ are the same in both cases, which implies

$$\begin{aligned} C_{\mathbf{D}^\times}(u_1,...,u_d) &= \sum_{j=0}^{d_2}(-1)^j\sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^*}(\mathbf{W}_{j,k}) \\ &= \sum_{j=0}^{d_2}(-1)^j\sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^+}(\mathbf{W}_{j,k}) \\ &= C_{\mathbf{D}^\circ}(u_1,...,u_d). \end{aligned}$$

In particular, note that $I_2(\mathbf{D}^*,\mathbf{D}^{*\complement}) = I_2(\mathbf{D},\mathbf{D}^{\complement}) = \{1,...,d\}$ for every $\mathbf{D}\in\Delta$. Then $C_{\mathbf{D}^*}\equiv C_{\mathbf{D}^{*\complement}}$ implies $C_{\mathbf{D}}\equiv C_{\mathbf{D}^{\complement}}$ for every $\mathbf{D}\in\Delta$.

(ii) (a) From proposition (1) we have

$$C_{\mathbf{D}^*}(u_1,...,u_d) = \sum_{j=0}^{d_2}(-1)^j\sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k}). \tag{2.20}$$

Consider $j\in\{0,...,d_2\}$ and $k\in\{1,...,\binom{d_2}{j}\}$. From the way it is defined, $\mathbf{W}_{j,k}$ contains u_i for every $i\in I_1(\mathbf{D}^*,\mathbf{D}^\circ)$. The exchangeability of $C_{\mathbf{D}^\circ}$ implies that $C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k})$ is exchangeable over $I_1(\mathbf{D}^*,\mathbf{D}^\circ)$. Hence, (2.20) implies that $C_{\mathbf{D}^*}$ is exchangeable over $I_1(\mathbf{D}^*,\mathbf{D}^\circ)$. Now, consider $j\in\{0,...,d_2\}$. Each $\mathbf{W}_{j,k}$, $k\in\{1,...,\binom{d_2}{j}\}$ considers a different subset of size j of $I_2(\mathbf{D}^*,\mathbf{D}^\circ)$. If we consider $\sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k})$, each element in $I_2(\mathbf{D}^*,\mathbf{D}^\circ)$ appears in the same way as the others. Hence, given that $C_{\mathbf{D}^\circ}$ is exchangeable, $\sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k})$ is exchangeable over $I_2(\mathbf{D}^*,\mathbf{D}^\circ)$. Equation (2.20) then implies that $C_{\mathbf{D}^*}$ is exchangeable over $I_2(\mathbf{D}^*,\mathbf{D}^\circ)$. In particular $C_{\mathbf{D}^{\circ\complement}}$ is exchangeable over $I_2(\mathbf{D}^\circ,\mathbf{D}^{\circ\complement}) = \{1,...,d\}$.

(ii) (b) Considering proposition (1), to avoid confusion, in this part of the proof, we denote with a superindex $*$ all the corresponding notations to express $C_{\mathbf{D}^*}$ in terms $C_{\mathbf{D}^\circ}$ and with a superindex $+$ all the notations to express $C_{\mathbf{D}^+}$ in terms of $C_{\mathbf{D}^\circ}$. From the hypothesis we know $d_2^+ = d_2^*$, so no superindex is used for this value.

Let PR be any permutation that satisfies the hypothesis. We denote $i' = PR(i)$ and $A' = \{PR(i)|i \in A\}$ with $A \subseteq \{1, ..., d\}$. From proposition (1), we have

$$C_{\mathbf{D}^+}(u_{1'}, ..., u_{d'}) = \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k}^{+(1)}), \tag{2.21}$$

with

$$W_{j,k,i}^{+(1)} = \begin{cases} u_{i'} & \text{if } i \in I_1^+ \\ 1 - u_{i'} & \text{if } i \in S_{j,k}^+ \\ 1 & \text{if } i \notin I_1^+ \cup S_{j,k}^+ \end{cases}$$

$i \in \{1, ..., d\}$, and $S_{j,k}^+$ is the k-th element of size j of I_2^+, $j \in \{0, ..., d_2\}$ and $k \in \{1, ..., \binom{d_2}{j}\}$. Given that $C_{\mathbf{D}^\circ}$ is exchangeable, we have that

$$C_{D^\circ}(\mathbf{W}_{j,k}^{+(1)}) = C_{D^\circ}(\mathbf{W}_{j,k}^{(2)}), \tag{2.22}$$

with

$$W_{j,k,i}^{(2)} = \begin{cases} u_i & \text{if } i \in I_1(\mathbf{D}^*, \mathbf{D}^\circ) \\ 1 - u_i & \text{if } i \in S_{j,k}^{+\prime} \\ 1 & \text{if } i \notin I_1(\mathbf{D}^*, \mathbf{D}^\circ) \cup S_{j,k}^{+\prime} \end{cases}$$

For each $k \in \{1, ..., \binom{d_2}{j}\}$, $S_{j,k}^{+\prime}$ is a different subset of size j of $I_2(D^*, D^\circ)$. Hence,

$$\sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k}^{(2)}) = \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k}^{*}), \tag{2.23}$$

for $j \in \{0, ..., d_2\}$. Proposition (1) and equations (2.21) to (2.23) imply the result.

(ii) (c) Note that $|I_2(\mathbf{D}^*, \mathbf{D}^\circ)| = \frac{d}{2} \implies |I_2(\mathbf{D}^*, \mathbf{D}^{\circ\complement})| = \frac{d}{2}$. Consider any permutation of $\{1, ..., d\}$, $PR(i) = i'$, that assigns to each element of $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$ an element of $I_2(\mathbf{D}^*, \mathbf{D}^{\circ\complement})$. Given that $C_{\mathbf{D}^*}$ is exchangeable we can use (ii)(b)

$$C_{\mathbf{D}^{\circ\complement}}(u_1, ..., u_d) = C_{\mathbf{D}^\circ}(u_{1'}, ..., u_{d'}).$$

Considering that $C_{\mathbf{D}^\circ}$ is exchangeable, this implies $C_{\mathbf{D}^\circ} \equiv C_{\mathbf{D}^{\circ\complement}}$. (i) then implies $C_{\mathbf{D}} \equiv C_{\mathbf{D}^\complement}$ for all $\mathbf{D} \in \Delta$.

□

Similar to a distributional copula (see McNeil et al. (2005), Proposition (5.6)), in the continuous case, all the associated copulas are also invariant under strictly increasing transformations. We state this in the following proposition:

Proposition 3. *Let T_1,..., T_d be strictly increasing functions,* $\mathbf{X} = (X_1, ..., X_d)$ *a random vector with corresponding distribution function and marginals,* $\mathbf{D}$ *a type of dependence and* $\mathbf{D}$*-copula* $C_{\mathbf{D}}$*. Then, in the continuous case,*

$$\widetilde{\mathbf{X}} = (T_1(X_1), ..., T_d(X_d))$$

also has the same corresponding $\mathbf{D}$*-copula* $C_{\mathbf{D}}$*.*

Proof. Consider the following properties of inverses of strictly increasing functions, distribution functions and their inverses (see McNeil et al. (2005) propositions (5.6), and (A.3 vii) and (viii)).

(a) $T_i^{\leftarrow} \circ T_i(x_i) = x_i$.

(b) $G \circ G^{\leftarrow}(x) = x$ for any univariate continuous distribution function G.

Let the tilde $\sim$ denote the probability functions of $\widetilde{\mathbf{X}} = (T_1(X_1), ..., T_d(X_d))$. From (a) it follows that

$$\widetilde{F}_i(u_i) = F_i \circ T_i^{\leftarrow}(u_i), \tag{2.24}$$

and

$$\overline{\widetilde{F}}_i(u_i) = \overline{F}_i \circ T_i^{\leftarrow}(u_i), \tag{2.25}$$

for $i \in \{1, ..., d\}$. Consider the concepts of Definition 4; equations (2.24) and (2.25) imply that the marginal

$$\widetilde{F}_{D_i,i}(u_i) = F_{D_i,i} \circ T_i^{\leftarrow}(u_i), \tag{2.26}$$

for $i \in \{1, ..., d\}$. Properties (a) and (b) imply that the inverse of the marginal is given by

$$\widetilde{F}_{D_i,i}^{\leftarrow}(u_i) = T_i \circ F_{D_i,i}^{\leftarrow}(u_i), \tag{2.27}$$

for $i \in \{1, ..., d\}$. Considering (2.26), (2.27) and (a) the following two events are equivalent in probability

$$\widetilde{\mathcal{B}}_i(\widetilde{F}^{\leftarrow}_{D_i,i}(u_i)) \overset{P}{\sim} \mathcal{B}_i(F^{\leftarrow}_{D_i,i}(u_i)), \tag{2.28}$$

for $i \in \{1, ..., d\}$. This and the uniqueness of the copula, implied by theorem (1), implies

$$\widetilde{C_{\mathbf{D}}}(u_1, ..., u_d) = C_{\mathbf{D}}(u_1, ..., u_d).$$

□

The associated copulas can also be used to analyse the dependence structure of random variables after applying strictly monotone transformations to their variables, for the bivariate version see Nelsen (1999), Theorem 2.4.4, p.26, and Embrechts et al. (2001), Theorem 2.7, p.6.

Proposition 4. *Let T_1,..., T_d be strictly monotone functions, $\mathbf{X} = (X_1, ..., X_d)$ a random vector with corresponding distributional copula C. Then the distributional copula of $\widetilde{\mathbf{X}} = (T_1(X_1), ..., T_d(X_d))$ is the associated $\mathbf{D}$-copula $C_{\mathbf{D}}$ of $\mathbf{X}$, with*

$$D_i = \begin{cases} L \text{ if } T_i \text{ is strictly increasing} \\ U \text{ if } T_i \text{ is strictly decreasing} \end{cases},$$

for $i \in \{1, .., d\}$, whose expression is given by Corollary (1).

Proof. Note that for monotone T_i it holds that $T_i^{\leftarrow} \circ T_i(x_i) = x_i$. Hence, we use (a) and (b) of the previous proof. Again, we use the tilde $\sim$ to denote the probability functions of $\widetilde{\mathbf{X}} = (T_1(X_1), ..., T_d(X_d))$. Just as in (2.24), we have that if T_i is strictly increasing, $\widetilde{F}_i(u_i) = F_i \circ T_i^{\leftarrow}(u_i)$. If T_i is strictly decreasing,

$$\widetilde{F}_i(u_i) = \overline{F}_i \circ T_i^{\leftarrow}(u_i). \tag{2.29}$$

Hence

$$\widetilde{F}_i^{\leftarrow}(u_i) = \begin{cases} T_i \circ F_i^{\leftarrow}(u_i) \text{ if } T_i \text{ is strictly increasing} \\ T_i \circ \overline{F}_i^{\leftarrow}(u_i) \text{ if } T_i \text{ is strictly decreasing} \end{cases},$$

for $i \in \{1, ..., d\}$. Using this, we have:

$$T_i(X_i) \leq (\widetilde{F}_i^{\leftarrow}(u_i)) \overset{P}{\sim} \mathcal{B}_i(F^{\leftarrow}_{D_i,i}(u_i)),$$

for $i \in \{1, ..., d\}$, which implies

$$\widetilde{C}(u_1, ..., u_d) = C_{\mathbf{D}}(u_1, ..., u_d).$$

□

2.2. Associated Tail Dependence Functions and Tail Dependence Coefficients.

Considering the results obtained so far, it is possible to introduce a general definition of tail dependence function and tail dependence coefficients considering the dependence $\mathbf{D}$. For the analysis of the conditions of the existence of the tail dependence function see Mikosch (2006). The general expression of the tail dependence function is the following (see Nikoloulopoulos et al. (2009))

Definition 7. *Let $I = \{1, ..., d\}$, $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with copula C, $\mathbf{D} = (D_1, ..., D_d)$ be a type of dependence and $C_{\mathbf{D}}$ be the copula of the random vector $\mathbf{V_D}$ of Definition 3. For any $\emptyset \neq S \subseteq I$, let $\mathbf{D}(S)$ denote the corresponding $|S|$-dimensional marginal dependence of $\mathbf{D}$ and $C_{\mathbf{D},S}$ the copula of the $|S|$-dimensional marginal $\{V_{D_i,i} | i \in S\}$. Define the associated $\mathbf{D}(S)$-tail dependence functions $b_{\mathbf{D},S}$ of $C_{\mathbf{D}}$, $\emptyset \neq S \subseteq I$ as*

$$b_{\mathbf{D},S}(w_i, i \in S) = \lim_{u \downarrow 0} \frac{C_{\mathbf{D},S}(uw_i,\ i \in S)}{u}, \forall w = (w_1, ..., w_d) \in \mathbb{R}^d_+.$$

Given that these functions come from the associated copulas, we call the set of all $\mathbf{D}$-tail dependence functions the associated tail dependence functions. When $S = \{1, ..., d\}$ we omit such subindex.

The tail dependence functions was introduced by Nikoloulopoulos et al. (2009) as a generalisation to the tail dependence coefficient introduced by Joe (1993) to determine the existence of dependence among random variables. With the definition of the general tail dependence coefficient, that we now present, it is possible to determine the existence of tail dependence for a general dependence $\mathbf{D}$.

Definition 8. *Consider the same conditions of Definition 7. Define the associated $\mathbf{D}(S)$-tail dependence coefficients $\lambda_{\mathbf{D}(S)}$ of $C_{\mathbf{D}}$, $\emptyset \neq S \subseteq I$ as*

$$\lambda_{\mathbf{D},S}(w_i,\ i \in S) = \lim_{u \downarrow 0} \frac{C_{\mathbf{D},S}(u, .., u)}{u}.$$

We say that $(\mathbf{D}, S)$*-tail dependence exists whenever* $\lambda_{\mathbf{D},S} > 0$.

3. Modelling General Dependence

In this section we analyse dependence in copula models. We analyse two examples, the perfect dependence cases and the elliptically contoured distributions. With this analysis, it is possible to know the general dependence and tail dependence structure implied by the use of these models. For the perfect dependence case we obtain the associated copulas of the perfect positive dependence model. We then prove that these copulas correspond to the use of strictly monotone transformations on a random variable, so we call this copulas the monotonic copulas. For the elliptically contoured distributions we prove a proposition that chacaterises their corresponding associated copulas. We then present the associated tail dependence functions of the Student-t copula model. This model accounts for all 2^d types of tail dependencies. The analysis of general dependence presented in this section complements the analysis of only positive tail dependence for these models.

3.1. Perfect Dependence Cases.

We now analyse the most basic examples of copula models. They correspond to all the variables being either independent or perfectly dependent. We first present the independent copula. We then present the associated copulas of the perfect positive dependence model and prove that they correspond to the use of strictly monotone transformations on a random variable. It follows that these are the copulas of the perfect dependence models.

Let $\mathbf{U} = (U_1, ..., U_d)$ be a random vector with $\{U_i\}_{i=1}^d$ independent uniformly distributed random variables. The distribution function of U is the copula $C(u_1, ..., u_d) = \prod_{i=1}^d u_i$, which is known as the independence copula. It follows that the associated copula are also equal to the independence copula. This is the copula of any random vector $X = (X_1, ..., X_d)$ with $\{X_i\}_{i=1}^d$ independent random variables.

Let $\mathbf{U}$ be the d-dimensional vector $\mathbf{U} = (W, ..., W)$ with W a uniform random variable. The distribution function of $\mathbf{U}$ is the copula

$$C(u_1, ..., u_d) = \min\{u_i\}_{i=1}^d. \tag{3.1}$$

We now analyse the general associated copula $C_{\mathbf{D}}$ for this example. Let $\mathbf{D}$ be a type of dependence and $I = \{1, ..., d\}$. Define $I_L = \{i \in I | D_i = L\}$ and $I_U = \{i \in I | D_i = L\}$. From Definition 3, the associated $\mathbf{D}$-copula $C_{\mathbf{D}}$ is the distribution function of the vector $\mathbf{V_D}$. Let us assume that neither I_L nor I_U are empty (the other two cases have just been analysed), then the associated $\mathbf{D}$-copula is

$$C_{\mathbf{D}}(u_1, ..., u_d) = P((W \leq \min\{u_i\}_{i \in I_L}) \cap W \geq \max\{1 - u_i\}_{i \in I_U}).$$

It follows that, for $\min\{u_i\}_{i \in I_L} > \max\{1 - u_i\}_{i \in I_U}$, this probability is equal to zero; in another case we have

$$C_{\mathbf{D}}(u_1, ..., u_d) = \min\{u_i\}_{i \in I_L} + \min\{u_i\}_{i \in I_U} - 1.$$

Therefore, a general expression is

$$C_{\mathbf{D}}(u_1, ..., u_d) = \max\{0, \min\{u_i\}_{i \in I_L} + \min\{u_i\}_{i \in I_U} - 1\}. \tag{3.2}$$

Note that $C_{\mathbf{D}} = C_{\mathbf{D}^{\complement}}$. Hence, the d-dimensional vector $\mathbf{U} = (W, ..., W)$ is complement symmetric, according to Definition 5. There are 2^{d-1} different associated copulas of this vector. In the bivariate case the associated (L, U)-copula C_{LU} is equal to the Fréchet lower bound copula, also known as the countermonotonicity copula.

The copulas obtained in (3.1) and (3.2) are a generalisation of the comonotonicity and the countermonotonicity copulas of the bivariate case. In the bivariate case, they are the Fréchet bounds for copula and correspond to the use of strictly monotone transformations on a random variable. In the following proposition we prove that, in d dimensions, the copulas of (3.1) and (3.2) also correspond to the use of strictly monotone transformations on a random variable. Because of this, we call these copulas the monotonic copulas.

Proposition 5. *Let Z be a random variable, and let $\{T_i\}_{i=1}^d$ be strictly monotone functions, then the distributional copula of the vector $X = (T_1(Z), ...T_d(Z))$ is one of the monotonic copulas of equations (3.1) or (3.2) with $\mathbf{D} = (D_1, ..., D_d)$,*

$$D_i = \begin{cases} L \text{ if } T_i \text{ is strictly increasing} \\ U \text{ if } T_i \text{ is strictly decreasing} \end{cases}.$$

Conversely, consider a random vector $\mathbf{X} = (X_1, ..., X_d)$ *whose distributional copula is a monotonic copula of equation (3.1) or (3.2) for certain* $\mathbf{D}$*. Then there exist monotone functions* $\{T_i\}_{i=1}^d$ *and a random variable* Z *such that*

$$(X_1, ..., X_d) \stackrel{d}{=} (T_1(Z), ..., T_d(Z)), \tag{3.3}$$

the $\{T_i\}_{i=1}^d$ *satisfy that* T_i *is strictly increasing if* $D_i = L$ *and strictly decreasing if* $D_i = U$ *for* $i \in \{1, ..., d\}$*. In both cases the vector* $\mathbf{X}$ *is complement symmetric.*

Proof. Let F be the distribution function of Z. Considering the uniform random variable $F(Z)$ it is clear that the copula of the d-dimensional vector $(Z, ..., Z)$ is the Fréchet upper bound copula $\min\{u_i\}_{i=1}^d$of equation (3.1). The result is then implied by proposition (4).

The converse statement is a generalisation of Embrechts et al. (2001), Theorem 3.1, p.10. We have that the distributional copula of $\mathbf{X}$ is a monotonic copula for certain $\mathbf{D}$. Note that the associated $\mathbf{D}$-copula of $\mathbf{X}$ is the Fréchet upper bound copula. Let $\{\alpha_i\}_{i=1}^d$ be any invertible monotone functions that satisfy α_i is strictly increasing if $D_i = L$ and strictly decreasing if $D_i = U$ for $i \in \{1, ..., d\}$. Proposition (4) implies that the copula of $\mathbf{A} = (\alpha_1(X_1), ..., \alpha_d(X_d))$ is the Fréchet upper bound copula. According to Fréchet (1951) and Embrechts et al. (2002), there exists a random variable Z and strictly increasing $\{\beta_i\}_{i=1}^d$ such that

$$(\alpha_1(X_1), ..., \alpha_d(X_d)) \stackrel{d}{=} (\beta_1(Z), ..., \beta_d(Z)).$$

By defining $T_i = \alpha_i^{-1} \circ \beta_i$ for $i \in \{1, ..., d\}$ we get the result.

In both cases the associated copulas of $\mathbf{X}$ are the monotonic copulas implying that the vector is complement symmetric. □

This means that the copula of a perfect dependent model, where all variables have perfect positive or negative dependence, is a monotonic copula.

3.2. **Elliptically Contoured Distributions.**

We now anlayse the dependence structure of elliptically contoured distributions. We first present the definition of this model. Then we present its corresponding associated copulas. Finally we present the associated tail dependence functions of the Student-t copula model.

Elliptically contoured distributions, or elliptical distributions, were introduced by Kelker (1970) and have been analysed by several authors (see for example Fang and Ng (1990), Gupta and Varga (1993)). They have the following form.

Definition 9. *The random vector* $\mathbf{X} = (X_1, ..., X_d)$ *has a multivariate elliptical distribution, denoted as* $\mathbf{X} \sim El_d(\mu, \Sigma, \psi)$, *if for* $\mathbf{x} = (x_1, ..., x_d)'$ *its characteristic function has the form*

$$\varphi(\mathbf{x}; \mu, \Sigma) = \exp(i\mathbf{x}'\mu)\psi_d(\frac{1}{2}\mathbf{x}'\Sigma\mathbf{x}),$$

with μ *a vector,* $\Sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ *a symmetric positive-definite matrix and* $\psi_d(t)$ *a function called the characteristic generator.*

They encompass a large number of distributions, see Valdez and Chernih (2003), Appendix. Several properties have been developed in the case when the joint density exists, see Gupta and Varga (1993) and Das Gupta et al. (1972). If it exists, the joint density $f(\mathbf{x};\boldsymbol{\mu},\Sigma)$ has the following form:

$$f(\mathbf{x};\boldsymbol{\mu},\Sigma) = c_d|\Sigma|^{-\frac{1}{2}} g_d\left(\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\right), \tag{3.4}$$

with $g_d(\cdot)$ a function called the density generator and c_d a normalising constant dependent of g_d (see Landsman and Valdez (2002)).

Elliptical distributions have been used in several areas including financial data analysis. In particular the Student-t distribution has been known to account for fat tails and tail dependence.

3.2.1. *The Associated Elliptically Contoured Copula.*

The copula of an elliptically contoured distribution is referred to as elliptically contoured copula or elliptical copula. This copula has been subject to numerous analysis, see for instance Fang et al. (2002), Abdous et al. (2005), Embrechts et al. (2001) or Demarta and McNeil (2005). One of the characteristics of elliptically contoured distributions is that their marginals $F_i(x)$ are also elliptically contoured with the same characteristic or density generator. If the d-dimensional copula density c exists the joint density f, the marginal densities f_i, the marginals F_i and the corresponding copula density satisfy the following relationship (see Fang et al. (2002)):

$$f(x_1, ..., x_d) = c(F_1(x_1), ..., F_d(x_d)) \times \prod_{i=1}^{d} f_n(x_n).$$

Note that the process of standardising the marginal distributions of X uses strictly increasing transformations. As stated in proposition (3), copulas are invariant under such transformations. This implies that the copulas associated to $\mathbf{X} \sim El_d(\mu, \Sigma, \psi)$ are the same as the copulas associated to $\mathbf{X}^* \sim El_d(0, R, \psi)$. Here $R = (\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{iI}\sigma_{jJ}}})_{1 \leq i,j \leq d}$ is the corresponding "correlation" matrix implied by the positive-definite matrix $\Sigma = (\sigma_{ij})_{1 \leq i,j \leq d}$ (see Embrechts et al. (2001), Theorem 5.2, p.23, Fang et al. (2002) and Demarta and McNeil (2005)). Because of this, for our study of elliptical copulas we assume $\mathbf{X} \sim El_d(R, \psi)$ with $R = (\rho_{ij})_{1 \leq i,j \leq d}$, which covers the more general case $\mathbf{X} \sim El_d(\mu, \Sigma, \psi)$.

Equations (2.1) and (2.4) imply that the associated copulas of $\mathbf{X}$ are determined by the joint distribution and the inverse of the marginal distributions. In general, there is no closed-form expression for the elliptical copula but it can be expressed in terms of multidimensional integrals of the joint density $f(\mathbf{x};R)$. This case covers a wide variety of distributions, see Valdez and Chernih (2003), Appendix. In the following proposition we prove an identity for the associated copula for this general case.

Proposition 6. *Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with multivariate elliptical distribution of Definition 9, with correlation matrix $R = (\rho_{ij})_{1 \leq i,j \leq d}$, that is $\mathbf{X} \sim El_d(R, \psi)$ and let $\mathbf{D}$ be a type of dependence. Then the associated $\mathbf{D}$-copula of $\mathbf{X}$ is the same as the distributional copula of $\mathbf{X}^+ \sim El_d(\wp R \wp, \psi)$, with $\wp$ a diagonal matrix (all values in it are zero except for the values in its diagonal) $\wp \in M_{d \times d}$, whose diagonal is $p = (p_1, ..., p_d)$ with*

$$p_i = \begin{cases} 1 & \text{if } D_i = L \\ -1 & \text{if } D_i = U \end{cases},$$

for $i \in \{1, ..., d\}$.

Proof. The vector $\wp\mathbf{X}$ is equal to $(T_1(X_1), ..., T_d(X_d))$ with $T_i(x) = p_i x$, $i \in \{1, ..., d\}$. Using Proposition (4), the distributional copula of $\wp\mathbf{X}$ is the associated $\mathbf{D}$-copula of $\mathbf{X}$. From the stochastic representation of $\mathbf{X}$ (see Fang and Ng (1990)), it follows that $\wp\mathbf{X} \sim El_d(\wp R \wp', \psi)$ (see Embrechts et al. (2001), Theorem 5.2). □

The symmetric nature of the elliptically contoured distributions and copula is well known. It follows from proposition (6) that elliptical copulas are complement symmetric.

Corollary 2. *Let* $\mathbf{X} = (X_1, ..., X_d)$ *be a random vector with multivariate elliptical distribution of Definition 9,* $\mathbf{X} \sim El_d(R, \psi)$*. Then* $\mathbf{X}$ *is complement symmetric according to Definition 5.*

Proof. Let $\mathbf{D}$ be a type of dependence and $\mathbf{D}^{\complement}$ the complement dependence of Definition 5. Denote $\wp_{\mathbf{D}}$ and $\wp_{\mathbf{D}^{\complement}}$ the corresponding diagonal matrices of proposition (6).

It is clear that $\wp_{\mathbf{D}^{\complement}} = -\wp_{\mathbf{D}}$, which implies

$$\wp_{\mathbf{D}^{\complement}} \cdot R \cdot \wp_{\mathbf{D}^{\complement}} = \wp_{\mathbf{D}} \cdot R \cdot \wp_{\mathbf{D}}.$$

Hence, both $C_{\mathbf{D}}$ and $C_{\mathbf{D}^{\complement}}$ are equal to the distributional copula of $\mathbf{X}^{+} \sim El_d(\wp_{\mathbf{D}} R \wp_{\mathbf{D}}, \psi)$. □

Proposition (6) makes possible to use the results regarding elliptical copulas in associated copulas. This also includes the analysis of tail dependence. In the bivariate case Klüppelberg et al. (2008) obtained an expression for the lower tail dependence function under regular variation conditions. The Gaussian copula does not account for lower tail dependence, proposition (6) implies that it does not account for tail dependence for all $\mathbf{D}$. In contrast the Student-t copula does account for tail dependence (see Joe (2011) and Nikoloulopoulos et al. (2009)). We now analyse this copula into more detail.

3.2.2. *The Multivariate Student-t Associated Tail Dependence Function.*

The Student-t copula is well known for accounting for stylised facts such as fat tail and the presence of tail dependence (see McNeil et al. (2005) and Demarta and McNeil (2005)). The Student-t copula with ν degrees of freedom and correlation matrix R is expressed in terms of integrals of its corresponding density $t_{\nu,R}$.

$$C(\mathbf{u}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^d |R|}} \left(1 + \frac{\mathbf{x}' R^{-1} \mathbf{x}}{\nu}\right)^{-\frac{\nu+d}{2}} d\mathbf{x},$$

with $\mathbf{u} = (u_1, ..., u_d)$ and $\mathbf{x} = (x_1, ..., x_d)'$.

Unlike the multivariate Gaussian distribution, the case of $R = I$ does not correspond to the independence case, see (Hult and Lindskog (2002)). Another interesting example is to consider perfect dependence. That is, the case when $R = (\rho_{ij})_{1 \le i,j \le d}$ satisfies $\rho_{ij} = 1$ if $i, j \in S_1$ or $i, j \in S_2$, and $\rho_{ij} = -1$ if $i \in S_1$, $j \in S_2$ or $i \in S_2$, $j \in S_1$, with S_1 and S_2 disjoint sets that satisfy $S_1 \cup S_2 = \{1, ..., d\}$. In this case $R = (\rho_{ij})_{1 \le i,j \le d}$ can be expressed as

$$R = \wp \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \wp$$

with $\wp \in M_{d \times d}$ a diagonal matrix, whose values in the diagonal are1 if $i \in S_1$ and -1 if $i \in S_2$, $i \in \{1, ..., d\}$. Inductively on d, it is straightforward to prove that the determinant of a matrix of ones is zero. It follows that $|R| = 0$, and hence the copula is not defined in this case. The Student-t copula is known for accounting for several types of tail dependence. McNeil et al. (2005) proved that, in the bivariate case, regardless of the value of the correlation coefficient ρ, the lower and upper tail dependence coefficients are positive. Nikoloulopoulos et al. (2009) analysed in full detail the extreme value properties of this copula and obtained an expression of the lower tail dependence function among other results. More recently, in the bivariate case, Joe (2011), p. 199, obtained an expression for the $\mathbf{D} = (L, U)$ and the $\mathbf{D} = (U, L)$ tail dependence coefficients proving that this copula accounts for negative tail dependence. In this subsection we present the expression for the associated $\mathbf{D}$-tail dependence function of the multivariate Student-t copula. Given that this function is positive for $|R| \neq 0$ and for all $\mathbf{D}$, the Student-t copula accounts for all types of tail dependence. This result follows from Nikoloulopoulos et al. (2009), Theorem 2.3, p.135 and proposition (6).

Proposition 7. *Let $\mathbf{X} = (X_1, ..., X_d)$ have multivariate t distribution with ν degrees of freedom, and correlation matrix $R = (\rho_{ij})_{1 \le i,j \le d}$, that is $\mathbf{X} \sim T_{d,\nu,R}$. Let $\mathbf{D} = (D_1, ..., D_d)$ be a type of dependence. Then the associated $\mathbf{D}$-tail dependence function $b_{\mathbf{D}}$ is given by*

$$b_{\mathbf{D}}(w) = \sum_{j=1}^{d} w_j T_{d-1,\nu+1,R'_j} \left(\sqrt{\frac{\nu+1}{1-\rho_{ij}^2}} \left[-\left(\frac{w_i}{w_j} \right)^{-\frac{1}{\nu}} + p_i p_j \rho_{ij} \right], i \in I_j \right),$$

with

$$R_j^* = \begin{pmatrix} 1 & \cdots & \rho^*_{1,j-1;j} & \rho^*_{1,j+1;j} & \cdots & \rho^*_{1,d;j} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \rho^*_{j-1,1;j} & \cdots & 1 & \rho^*_{j-1,j+1;j} & \cdots & \rho^*_{j-1,d;j} \\ \rho^*_{j+1,1;j} & \cdots & \rho^*_{j+1,j-1;j} & 1 & \cdots & \rho^*_{j+1,j-1;j} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^*_{d,1;j} & \cdots & \rho^*_{d,j-1;j} & \rho^*_{d,j+1;j} & \cdots & 1 \end{pmatrix};$$

$\rho^*_{i,k;j} = p_i p_k \frac{\rho_{ik}-\rho_{ij}\rho_{kj}}{\sqrt{1-\rho_{ij}^2}\sqrt{1-\rho_{kj}^2}}$, *the modified partial correlations;* $I_j = I - \{j\}$ *and*

$$p_j = \begin{cases} 1 & \text{if } D_j = L \\ -1 & \text{if } D_j = U \end{cases},$$

for $j \in \{1, ..., d\}$.

Proof. The definition presented in this work for the tail dependence functions has the same form for different dependencies. The only difference is the underlying associated copula. Proposition (6) then implies that the associated $\mathbf{D}$-tail dependence function of the random vector $\mathbf{X} \sim T_{d,\nu,R}$ is the lower tail dependence function of the vector $\mathbf{X}^+ \sim T_{d,\nu,\wp R \wp}$. $\wp$ is the diagonal matrix, whose diagonal is $p = (p_1, ..., p_d)$ with

$$p_i = \begin{cases} 1 & \text{if } D_i = L \\ -1 & \text{if } D_i = U \end{cases}$$

for $i \in \{1, ..., d\}$.

The modified correlation matrix is $\wp R \wp = R^* = (\rho^*_{ij})_{1 \le i,j \le d}$, it follows that

$$(\rho^*_{ij})_{1 \le i,j \le d} = (p_i p_j \rho_{ij})_{1 \le i,j \le d}.$$

Hence $(\rho^*_{ij})^2 = p_i^2 p_j^2 \rho_{ij} = 1 \cdot 1 \cdot \rho_{ij}^2 = \rho_{ij}^2$ Under this change, the partial correlations are modified as follows

$$\rho^*_{i,k;j} = p_i p_k \frac{\rho_{ik} - \rho_{ij}\rho_{kj}}{\sqrt{1-\rho_{ij}^2}\sqrt{1-\rho_{kj}^2}}.$$

The result is then implied by Nikoloulopoulos et al. (2009), Theorem 2.3, p.135. □

4. **Conclusions and Future Work**

In this paper we introduce the concepts to analyse, in the multivariate case, the whole dependence structure among random variables. For this, we present the **D**-probability functions, associated tail dependence functions and associated TDCs. This complements the use of the distributional and survival copulas C and $\widehat{C}$ and the lower and upper tail dependence functions and TDCs to analyse positive dependence.

We first obtain a number of theoretical results regarding the associated copulas introduced by Joe (1997). We present a version of Sklar theorem that links **D**-probability functions with the associated copulas. Together with the distributional and the survival copulas, the other associated copulas characterise the dependence structure among random variables. With them it is possible to analyse all types of dependence and tail dependence. In the bivariate case, this includes positive and negative dependence. If general dependence is not considered, the analysis of dependence structure among random variables is not complete.

We present an expression to link all the associated copulas of a random vector. After this we prove that they are invariant under strictly increasing transformations and characterise the copula of a vector after using monotone transformations. We then introduce the associated tail dependence functions and associated tail dependence coefficients of a random vector to analyse its general tail dependence.

We use the concepts and results obtained in the first part of the paper to analyse two examples of copula models. The first example corresponds to the perfect dependence cases. The corresponding copulas are a generalisation of the Fréchet copula bounds of the bivariate case, they correspond to the use of strictly monotone transformations on a random variable. Accordingly, we name these copulas the monotonic copulas. These copulas are all radial symmetric. The second example corresponds to the elliptical contoured distributions. For this example, we also obtain an expression for the corresponding associated copulas and prove that they are radial symmetric. As expected the Gaussian copula does not account for any type of tail dependence, regardless of the correlation matrix. Using a result by Nikoloulopoulos et al. (2009), we present an expression for the associated tail dependence function of the Student-t copula.

This result proves that this copula model accounts for all types of tail dependence, as long as its correlation matrix is non-singular.

The results obtained in this work allow us to understand better the dependence structure of a multivariate random vector. It is then possible to know more about the underlying assumptions of different copula models. Without analysing general dependence, the analysis of tail dependence in these models is therefore incomplete.

The Student-t has proven to be a better copula model than the Gaussian when modelling financial data. It is well known that this data has heavy tails and extreme dependencies and the assumption of only positive tail dependence has proven to be unrealistic. It is not surprising, but yet interesting that the Student-t copula accounts for extreme dependencies of all types. An application of non-positive tail dependence analysis appears in the context of hedging strategies. This tail dependence will minimise the risks and variability of the portfolio in times of economic crisis when extreme values are likely to appear.

With the formulas presented for the associated copulas, it is possible to extend the analysis to other copula models with closed-form expressions. This includes copulas such as the Archimedean, the Marshall-Ollkin and other models based on Laplace transforms. It must be borne in mind that the fact that a particular model does not account for a type of tail dependence does not mean it can not be used to model it. As long as a model accounts for one type of tail dependence, it can be used to model an arbitrary type of dependence. For example the multivariate Marshall-Olkin copula accounts only for lower tail dependence. However, if we want to model $\mathbf{D}$-tail dependence we can assume that the $\mathbf{D}$-copula is the Marshall-Olkin copula. Other interesting examples of copula models are the Vine copulas, the use of these copulas has proven to provide a flexible approach to tail dependence and account for asymmetric positive tail dependence (see for instance Nikoloulopoulos et al. (2012) or Joe et al. (2010)). These examples of the uses of the general dependence $\mathbf{D}$ are worth being studied in future research.

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