Centre for Computational Finance and Economic Agents

Working Paper Series

WP065-12

Alexander Guarin, Xiaoquan Liu and Wing Lon Ng

Valuation of American Options with Meshfree Methods

2012

University of Essex

CCFEA

www.ccfea.net

Valuation of American Options with Meshfree Methods

Alexander Guarin* Xiaoquan Liu† Wing Lon Ng‡

*Centre for Computational Finance and Economic Agents, University of Essex, Colchester, CO4 3SQ, UK and Banco de la República, Carrera 7 No. 14-78, Bogotá, Colombia. Email: aguarilo@banrep.gov.co.

†Corresponding author. Essex Finance Centre and Essex Business School, University of Essex, Colchester, CO4 3SQ, UK. Email: liux@essex.ac.uk. Phone: +44(0)1206 873849.

‡Centre for Computational Finance and Economic Agents, University of Essex, Colchester, UK. Email: wlng@essex.ac.uk. Phone: +44(0)1206 874684.

Valuation of American Options with Meshfree Methods

Abstract

In this paper, we price American options using the radial basis function (RBF) interpolation method. Two processes for the volatility are assumed: local volatility and stochastic volatility. In particular, we focus on the constant elasticity of variance (CEV) model (Cox and Ross (1976)) and the Heston model (Heston (1993)). Several experiments are performed to evaluate the pricing accuracy and the computational efficiency of the RBF method. The results are compared against solutions obtained by two traditional techniques in finance, namely the standard finite difference method (FDM) and the Monte Carlo simulation (MCS). The option prices approximated by the RBF interpolation are also contrasted with the results reported in other recent studies. The findings show that the RBF interpolation provides accurate and efficient option prices under the two volatility processes under investigation. In particular, the gains of using the meshfree method are observed in the Heston model due to its two-dimensional setting. Under the CEV model, the performance of the RBF method is similar to the FDM, but superior compared with the MCS. Under the Heston model specification, the RBF method outperforms both the FDM and the MCS.

Keywords: Meshfree Methods, Radial Basis Function Interpolation, American Options, Option Pricing, Heston Model, CEV Model.

JEL Codes: C63, C65, G12, G13

1 Introduction

The popular Black-Scholes (BS) model establishes the basis of modern option pricing theory (Black and Scholes (1973) and Merton (1973)). However, this model assumes that the volatility of the underlying asset returns is constant, which is not consistent with the empirical evidence (see Rubinstein (1994), Fouque et al. (2000) and Cont (2001)). In fact, the volatility is a function of the strike price (volatility smile) and the time to maturity of the contract (Dell'Era (2010)). In addition, the BS model does not capture the volatility clustering and the leverage effect (Fatone et al. (2009) and Mitra (2010)).

In order to relax the constant volatility assumption, researchers have proposed local and stochastic volatility models, combinations of them and other new features such as stochastic interest rates and stochastic jumps (Bakshi et al. (1997) and AitSahlia et al. (2010)). Local and stochastic volatility models have provided a valuable improvement in the option pricing theory and represent a useful instrument for practitioners in finance. In particular, stochastic volatility models have shown their ability to describe the observed data in the market and overcome the drawbacks exhibited in the BS model (Mitra (2010)). They use sophisticated processes to describe the dynamics of both the underlying asset and its variance (Fatone et al. (2009)).

However, in many cases closed-form solutions do not exist, or have to be approximated (Zhang and Lim (2006) and Chang et al. (2007)), or they become cumbersome, time-consuming or difficult to compute (Mitra (2010) and Park and Kim (2011)). Another important topic in option pricing is the valuation of American options, whose early exercise feature requires special treatment. In this case, it is not possible to compute the solution by an exact formula (Ikonen and Toivanen (2008), Vidal (2009) and O'Sullivan and O'Sullivan (2010)).

Researchers have dealt with the issues highlighted above using numerical schemes based on the Monte-Carlo simulations (MCS), the finite difference methods (FDM) and other mesh-based techniques. Recent examples include Boyle et al. (2003), Ikonen and Toivanen (2007), Ikonen and Toivanen (2008), Persson and Sydow (2010), AitSahlia et al. (2010), Düring and Fournié (2010) and O'Sullivan and O'Sullivan (2010). However, the MCS method is time-consuming and its application to price American options is not direct (see also Duck et al. (2005), Dutt and Welke (2008), and Liu (2010)). On the other hand, mesh-based methods such trees (Chen et al. (2002), Widdicks et al. (2002) and Beliaeva and Nawalkha (2010)) or FDM (Koc et al. (2003), Daum and Krichman (2006) and Duffy (2006)) face difficulties in complex and high-dimensional applications.

As a first contribution in this paper, we employ the radial basis function (RBF) interpolation from meshfree methods to obtain accurate, robust and efficient prices for American options under two specifications for the volatility, namely the constant elasticity of variance (CEV) model and the Heston model. Hence, we deal with the two problems highlighted above: first, the pricing of options under stochastic volatility, and second, the pricing of American options.

Meshfree methods arise as novel numerical approximation techniques that overcome some weaknesses faced by mesh-based methods (Duffy (2006)). They have been employed in fields such as computer graphics, non-uniform sampling, artificial intelligence, neural networks, data mining, signal processing, optimization and nanotechnology (see Liu (2003) and Fasshauer (2007) for more details). Mei and Cheng (2008) and Kelly (2009) reported that although the RBF methods have been applied successfully in many areas, there are only a handful of applications in finance, mainly in option pricing and usually only in a BS setting (e.g Hon and Mao (1999), Hon (2002), Koc et al. (2003), Fasshauer et al. (2004), Pettersson et al. (2008), Fasshauer et al. (2008) and Larsson et al. (2008)). This study extends the findings in the existing literature on the RBF methods in option pricing by concentrating on the CEV and Heston model.

As a second contribution, we perform several experiments to compare the option prices computed by the RBF interpolation with those obtained from the MCS and the FDM. American option prices under the Heston model are also compared with results reported in recent studies in the literature. The performance of the numerical techniques is evaluated in terms of accuracy and computational efficiency.

The results show that the RBF interpolation is a robust and efficient method for pricing American options. The method provides accurate option prices with small computation time in comparison with the other techniques and recent references in the literature. In addition, under this approach, few steps for discretization over time are required to obtain solutions converging to the benchmark. For the CEV model, a one-dimensional specification, the meshfree method outperforms the MCS and its performance is similar to the FDM. For the Heston model, a two-dimensional specification, the RBF interpolation method provides superior results compared with the FDM and the MCS.

The structure of the paper is organized as follows. Section 2 first reviews the selected option pricing models. In Section 3, we introduce the meshfree methods and illustrate the use of the RBF interpolation to compute the option prices under the CEV and Heston models. Section 4 reports and compares the results of the numerical experiments. Finally, Section 5 concludes.

2 Option Pricing Models

This section introduces the CEV model (Section 2.1) and the Heston model (Section 2.2). In particular, we discuss the stochastic process, the partial differential equation (PDE), and the boundaries conditions for each model.

2.1 The CEV Model

The CEV model is proposed in Cox and Ross (1976). It is a local volatility model, which assumes that the volatility is a function of the stock price and the time. It is able to fit implied volatility surfaces and account for the existence of the leverage effect (Boyle and Tian (1999), Vidal (2009) and Mitra (2010)). However, the volatility is assumed to be perfectly correlated with the stock price S and the model is not able to capture volatility clustering (Mitra (2010)).

Recent applications of the CEV model include the pricing of European, barrier and lookback options (Park and Kim (2011)), American options in defaultable equity (Vidal (2009)), and options on the S&P 500 index (Chen et al. (2009)).

The CEV model assumes that the dynamics of the underlying stock price $S(t)$ at time t under the $\mathcal{Q}$-measure is modelled by a stochastic differential equation (SDE)

$$dS(t) = (\mathtt{r} - q)\, S(t)\, dt + \tilde{\sigma} S(t)^{\frac{\beta}{2}}\, dW(t)\,, \tag{1}$$

where $\mathtt{r}$, q, $\tilde{\sigma}$ and β are parameters for the risk-free interest rate, the dividend yield, the volatility, and the elasticity of variance with respect to the stock price, respectively. The variable $W(t)$ denotes a Wiener process.

For different values of β the CEV model can be reduced to a number of popular option pricing models. For instance, if $\beta = 2$, the CEV model becomes the BS model; if $\beta = 1$, the CEV model is transformed into the square-root model (see Chen et al. (2009) and Choi et al. (2010)).

Consider $\mathcal{U} = \mathcal{U}(S, \mathtt{t})$ as the option price under the CEV model with stock price S and time left to maturity t. The latter is defined as $\mathsf{t} = \mathsf{T} - t$, where T is the maturity date and t is the current time. The option price $\mathcal{U}(S, \mathsf{t})$ satisfies the PDE

$$\frac{\partial \mathcal{U}}{\partial \mathsf{t}} + \frac{1}{2}\tilde{\sigma}^2 S^\beta \frac{\partial^2 \mathcal{U}}{\partial S^2} + (\mathtt{r} - q)\, S \frac{\partial \mathcal{U}}{\partial S} - \mathtt{r}\mathcal{U} = 0. \tag{2}$$

The American put option is subject to the conditions (Wong and Zhao (2008))

$$\mathcal{U}(S, 0) = max\,(E - S, 0)\,, \quad 0 < S < \infty, \tag{3a}$$

$$\mathcal{U}(S, \mathsf{t}) = E \exp^{-(\mathtt{r}-q)\mathsf{t}}, \quad S = 0,\ \mathsf{t} > 0, \tag{3b}$$

$$\mathcal{U}(S, \mathsf{t}) \to 0, \quad S \to \infty,\ \mathsf{t} > 0, \tag{3c}$$

$$\mathcal{U}(S, \mathsf{t}) \geq max\,(E - S, 0)\,, \quad 0 < S < \infty,\ \mathsf{t} > 0, \tag{3d}$$

where E is the strike price (i.e. exercise price). For American call options, the set of constraints is defined as

$$\mathcal{U}(S, 0) = max\,(S - E, 0)\,, \quad 0 < S < \infty, \tag{4a}$$

$$\mathcal{U}(S, \mathsf{t}) = 0, \quad S = 0,\ \mathsf{t} > 0, \tag{4b}$$

$$\mathcal{U}(S, \mathsf{t}) \to S, \quad S \to \infty,\ \mathsf{t} > 0, \tag{4c}$$

$$\mathcal{U}(S, \mathsf{t}) \geq max\,(S - E, 0)\,, \quad 0 < S < \infty,\ \mathsf{t} > 0. \tag{4d}$$

For a put option, equation (3a) corresponds to the payoff at the maturity date. Equations (3b-3c) define the boundary conditions of the problem. This set of constraints establishes the European option pricing problem. Equation (3d) is the early exercise condition. It means that the exercise is permitted at any time $\mathsf{t} > 0$ during the life of the option $[0, \mathsf{T}]$. This condition must be added to the previous constraints for pricing American options. The same definitions apply to equations (4a-4d) in the case of a call option.

Special Case : the CEV Model with $\beta = 2$

With $\beta = 2$, the CEV model becomes the BS model. In this particular case the underlying asset price $S(t)$ at time t under the $\mathcal{Q}$-measure follows the stochastic differential equation

$$dS(t) = (\mathtt{r} - q)\, S(t)\, dt + \tilde{\sigma} S(t)\, dW(t)\,, \tag{5}$$

where $\mathtt{r}$ is the risk-free interest rate, q is the dividend yield, $\tilde{\sigma}$ is the constant volatility, and $W(t)$ is a Wiener process.

Under this framework, the option price $\mathcal{U}(S, \mathfrak{t})$ satisfies the time-dependent linear PDE

$$\frac{\partial \mathcal{U}}{\partial \mathfrak{t}} + \frac{1}{2}\tilde{\sigma}^2 S^2 \frac{\partial^2 \mathcal{U}}{\partial S^2} + (\mathtt{r} - q) S \frac{\partial \mathcal{U}}{\partial S} - \mathtt{r}\mathcal{U} = 0. \tag{6}$$

The American put option is subject to the conditions (3a-3d). For American call options, the set of constraints is given by equations (4a-4d). The pricing of European put and call options does not consider the early exercise condition (3d) and (4d), respectively.

2.2 The Heston Model

Heston (1993) proposes a stochastic volatility model, which has been widely adopted in the option pricing literature. Stochastic volatility models allow for more flexible dynamics for the volatility process. Nevertheless, these models tend to be analytically less tractable. In fact, it is common that they have no closed-form solution to compute option prices. Therefore, numerical methods are required to evaluate the options.

Literature on applications of the Heston model in option pricing includes to Zhang and Shu (2003) who value S&P 500 index options. In addition, Ikonen and Toivanen (2007), Ikonen and Toivanen (2008) and O'Sullivan and O'Sullivan (2010) price American options and compare a number of numerical methods. Moreover, Fatone et al. (2009) price European vanilla options using a multi-scale model and AitSahlia et al. (2010) perform pricing and hedging on S&P 100 index option data.

The Heston model under the $\mathcal{Q}$-measure is given by

$$dS(t) = (\mathtt{r} - q) S(t) dt + \sqrt{V(t)} S(t) dW_1(t), \tag{7}$$

$$dV(t) = \kappa(\theta - V(t)) dt + \sigma \sqrt{V(t)} dW_2(t), \tag{8}$$

and

$$Corr[dW_1(t), dW_2(t)] = \varrho dt, \tag{9}$$

where $S(t)$ and $V(t)$ are the asset price and variance at time t, respectively. The parameter $\mathtt{r}$ is the risk-free rate and q is the dividend yield. The variance $V(t)$ is modelled as a square-root mean reverting process (Cox et al. (1985)), where κ is mean reversion speed, θ is the long-run mean and σ is the volatility of the variance. The variables $W_i(t)$ for $i = 1, 2$ are

Wiener processes with correlation ϱ.

Consider $U = U(S, V, t)$ as the option price under Heston model at current time t, stock price S and variance V. Heston (1993) shows that $U = U(S, V, t)$ can be obtained by solving a two-dimensional parabolic PDE

$$\frac{\partial U}{\partial t} + \frac{1}{2} S^2 V \frac{\partial^2 U}{\partial S^2} + \varrho \sigma V S \frac{\partial^2 U}{\partial S \partial V} + \frac{1}{2} \sigma^2 V \frac{\partial^2 U}{\partial V^2} + (\mathtt{r} - q) S \frac{\partial U}{\partial S} + \left[\kappa(\theta - V) - \zeta V\right] \frac{\partial U}{\partial V} - \mathtt{r} U = 0, \quad (10)$$

subject to a final condition and boundary conditions. The constant parameter ζ is the market price of risk. Similar to Oosterlee (2003), Ikonen and Toivanen (2007), Ikonen and Toivanen (2008) and Persson and Sydow (2010), it is assumed that $\zeta = 0$.

Under the transformation of variables $\mathcal{X} = ln(S/E)$, $\tilde{U} = \tilde{U}(\mathcal{X}, V, \mathsf{t})$ and $\mathsf{t} = \mathsf{T} - t$, where E is the strike price, t is the remaining time to maturity and T is the maturity date, the PDE (10) can be written as

$$\frac{\partial \tilde{U}}{\partial \mathsf{t}} + \frac{1}{2} V \frac{\partial^2 \tilde{U}}{\partial \mathcal{X}^2} + \varrho \sigma V \frac{\partial^2 \tilde{U}}{\partial \mathcal{X} \partial V} + \frac{1}{2} \sigma^2 V \frac{\partial^2 \tilde{U}}{\partial V^2} + \left((\mathtt{r} - q) - \frac{1}{2} V\right) \frac{\partial \tilde{U}}{\partial \mathcal{X}} + \left[\kappa(\theta - V)\right] \frac{\partial \tilde{U}}{\partial V} - \mathtt{r} \tilde{U} = 0, \quad (11)$$

to be solved on $\mathbb{R} \times \mathbb{R}^+$.

The price of an American put option is given by the solution of the PDE (11) subject to the set of constraints (Düring and Fournié (2010))

$$\tilde{U}(\mathcal{X}, V, 0) = max\left(1 - exp^{\mathcal{X}}, 0\right), \qquad \mathcal{X} \in \mathbb{R},\ V > 0, \quad (12a)$$

$$\tilde{U}(\mathcal{X}, V, \mathsf{t}) \to 1, \qquad \mathcal{X} \to -\infty,\ V > 0,\ \mathsf{t} > 0, \quad (12b)$$

$$\frac{\partial \tilde{U}(\mathcal{X}, V, \mathsf{t})}{\partial V} \to 0, \qquad \mathcal{X} \in \mathbb{R},\ V \to \infty,\ \mathsf{t} > 0, \quad (12c)$$

$$\tilde{U}(\mathcal{X}, V, \mathsf{t}) \to 0, \qquad \mathcal{X} \to +\infty,\ V > 0,\ \mathsf{t} > 0, \quad (12d)$$

$$\frac{\partial \tilde{U}(\mathcal{X}, V, \mathsf{t})}{\partial V} \to 0, \qquad \mathcal{X} \in \mathbb{R},\ V \to 0,\ \mathsf{t} > 0, \quad (12e)$$

$$\tilde{U}(\mathcal{X}, V, \mathsf{t}) \geq max\left(1 - exp^{\mathcal{X}}, 0\right), \qquad \mathcal{X} \in \mathbb{R},\ V > 0,\ \mathsf{t} > 0, \quad (12f)$$

In the case of American call options, the set of constraints is given by

$$\tilde{U}(\mathcal{X}, V, 0) = max\left(\exp^{\mathcal{X}} - 1, 0\right), \qquad \mathcal{X} \in \mathbb{R},\ V > 0, \tag{13a}$$

$$\tilde{U}(\mathcal{X}, V, \mathfrak{t}) \to 0, \qquad \mathcal{X} \to -\infty,\ V > 0,\ \mathfrak{t} > 0, \tag{13b}$$

$$\frac{\partial \tilde{U}(\mathcal{X}, V, \mathfrak{t})}{\partial V} \to 0, \qquad \mathcal{X} \in \mathbb{R},\ V \to \infty,\ \mathfrak{t} > 0, \tag{13c}$$

$$\tilde{U}(\mathcal{X}, V, \mathfrak{t}) \to 1, \qquad \mathcal{X} \to +\infty,\ V > 0,\ \mathfrak{t} > 0, \tag{13d}$$

$$\frac{\partial \tilde{U}(\mathcal{X}, V, \mathfrak{t})}{\partial V} \to 0, \qquad \mathcal{X} \in \mathbb{R},\ V \to 0,\ \mathfrak{t} > 0, \tag{13e}$$

$$\tilde{U}(\mathcal{X}, V, \mathfrak{t}) \geq max\left(\exp^{\mathcal{X}} - 1, 0\right), \qquad \mathcal{X} \in \mathbb{R},\ V > 0,\ \mathfrak{t} > 0. \tag{13f}$$

Equations (12a) and (13a) correspond to the payoff at the maturity date, while equations (12b-12e) and (13b-13e) define the boundary conditions for the put and call options, respectively.

The pricing of American options includes an additional feature of early exercise condition. It means that the exercise is permitted at any time $\mathfrak{t} > 0$ during the life of the option $[0, \mathsf{T}]$. There is no exact formula when the early exercise condition is involved. So it is necessary to impose the constraints defined in equations (12f) and (13f) for put and call options, respectively.

3 Meshfree Methods

In this section, we introduce meshfree approximation methods and in particular the radial basis function (RBF) interpolation method.

Mesh-based methods are the traditional approach in finance to deal with the numerical approximation of PDEs. These methods discretize the spatial domain using an underlying computational mesh or grid with nodes providing a predefined relationship between them (Liu (2003)). These methods are applied mostly to one- and two-dimensional applications (Fasshauer (2007)). Within these techniques, the finite difference method (FDM) is the most common numerical tool used by practitioners in finance (Duffy (2006)). They work well in simple and low-dimensional cases. However, mesh-based methods face difficulties in more complex applications. The drawbacks are associated with the building of the underlying mesh, the discretization, its regularity conditions and the time-consuming requirement to implement it in multi-dimensional problems (Fasshauer (2007)). In fact, the computational complexity in the construction of a fixed grid grows exponentially with the dimension, becoming a difficult task in two or more dimensions (Duffy (2006), Daum (2005) and Daum and Krichman (2006)). Besides, the FDM achieves low-order polynomial accuracy and suffers from oscillation problems (Koc et al. (2003) and Duffy (2006)).

Unlike mesh-based approaches, meshfree methods do not require the use of an underlying grid with connectivity among its knots. Instead, these methods are based on a set of independent nodes, which are scattered on the domain of the problem (Liu (2003) and Li and Liu (2004)). Although there is no relationship among nodes, these are used to establish a system of equations for the whole domain. Hence, meshfree techniques are able to deal with issues where the use of fixed and regular meshes are a drawback (Liu (2003) and Liu and Gu (2005)). Meshfree methods are adaptive and versatile approximation techniques for the study of problems with complex geometries and irregular discretization (Liu (2003) and Fasshauer (2007)). In fact, as there is no mesh, these methods are relatively easy to implement in multi-dimensional problems (Duffy (2006)).

Liu (2003) points out that the accuracy and efficiency in the numerical approximation will depend on the number of points and their distribution on the space of the problem. Nevertheless, these two elements can be modified to improve the results. As there is no predefined relationship between the nodes, they can be added on or removed from different areas depending on the needs of the study. Liu (2003) also states that the allocation of points can be generated automatically such that the implicit costs in the creation of the mesh are eliminated.

The most popular meshfree techniques include the RBF interpolation method, the moving least squares (MLS) approximation method, the smooth particle hydrodynamics method, the element-free Galerkin method, the meshless local Petrov-Galerkin method, the reproducing kernel particle method, the diffuse element method and the partition of unity finite element method. Iske (2004) and Fasshauer (2006, 2007) are excellent references on the RBF interpolation and the MLS approximation. Liu (2003) and Li and Liu (2004) provide a very good introduction to the other techniques. Liu and Liu (2003) present a historical review on the meshfree methods in general, while Fasshauer (2007) focuses on the RBF techniques.

In this paper, we focus on the RBF interpolation. There exist in engineering many applications of the RBF interpolation method addressed to the solution of PDEs (see Liu (2003) and Fasshauer (2007)). However, in finance the number of RBF applications is rather small. They are concentrated in the solution of time-dependent PDEs for pricing options under the Black-Scholes (BS) model. For example, Hon and Mao (1999), Koc et al. (2003) and Kelly (2009) apply the RBF interpolation to solve the BS PDE for pricing European and American options (see also Goto et al. (2007), who considered Asian Options, also in a BS setting). Fasshauer et al. (2004) employ the RBF approach to evaluate multi-asset American options. Pettersson et al. (2008) derive a RBF method to price European basket call options. Fasshauer et al. (2008) consider the RBF approach for pricing options with non-smooth payoffs, in particular American digital options. Hon (2002) and Mei and Cheng (2008) perform quasi-interpolation and RBF approximation for pricing American and European options, respectively. Larsson et al. (2008) use the generalized Fourier transform along with RBF method for pricing multi-asset options.

In the following, we first introduce the RBF interpolation method (Section 3.1). We then explain the approximation of option prices using the RBF method under the CEV model (Section 3.2) and the Heston model (Section 3.3)

3.1 Radial Basis Function Interpolation

The RBF interpolation deals with univariate basis functions and a specific norm (commonly the Euclidean norm) to reduce a multi-dimensional problem into a one-dimensional issue (Fasshauer (2006, 2007)). In fact, Koc et al. (2003) and Duffy (2006) state that the RBF method is independent of the dimension of the problem. Hence, the approach deals with high-dimensional data with relative ease and its numerical results offer an efficient, highly accurate and versatile spatial approximation to the true solution (Duffy (2006)). In addition, the technique accounts for correlation terms without requiring special development (Fasshauer et al. (2004)). This feature is of crucial importance in the growing market of multi-asset derivative products.

Fasshauer (2006, 2007) explain that the RBF interpolation method approximates the value of a function as the weighted sum of RBFs. These functions are evaluated on a set of points called *centers*, which are quasi-randomly scattered over the domain of the problem. The weights are found by matching the approximated and observed values of the function. Once the interpolation weights are computed, they are used to estimate the value of the function at any point over entire domain.

Following Fasshauer (2007), we consider the set of centers $\mathbf{Z} = [\mathbf{z}_1, \ldots, \mathbf{z}_K]'$ with $\mathbf{z}_k \in$

$\mathbb{R}^d$, $d \geq 1$ and the data values $g_k \in \mathbb{R}$. We assume that

$$g_k = f\left(\mathbf{z}_k, t\right), \quad k = 1, \ldots, K,$$

where f is an unknown function and t is the time. We also define $f\left(\mathbf{Z}, t\right)$ as a linear combination of K certain basic functions

$$f\left(\mathbf{Z}, t\right) \simeq \sum_{k=1}^{K} \delta_k\left(t\right) \varphi\left(\| \mathbf{Z} - \mathbf{z}_k \|\right), \quad k = 1, \ldots, K, \tag{14}$$

where the coefficients $\delta_k\left(t\right)$ are the unknown weights, $\varphi\left(\cdot\right)$ is the chosen radial basis function and $\| \cdot \|$ is the Euclidean norm. Fasshauer (2007) shows that equation (14) is basically a system of linear equations

$$\begin{bmatrix} f\left(\mathbf{z}_1, t\right) \\ f\left(\mathbf{z}_2, t\right) \\ \vdots \\ f\left(\mathbf{z}_K, t\right) \end{bmatrix} \simeq \begin{bmatrix} \varphi\left(\| \mathbf{z}_1 - \mathbf{z}_1 \|\right) & \varphi\left(\| \mathbf{z}_1 - \mathbf{z}_2 \|\right) & \ldots & \varphi\left(\| \mathbf{z}_1 - \mathbf{z}_K \|\right) \\ \varphi\left(\| \mathbf{z}_2 - \mathbf{z}_1 \|\right) & \varphi\left(\| \mathbf{z}_2 - \mathbf{z}_2 \|\right) & \ldots & \varphi\left(\| \mathbf{z}_2 - \mathbf{z}_K \|\right) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi\left(\| \mathbf{z}_K - \mathbf{z}_1 \|\right) & \varphi\left(\| \mathbf{z}_K - \mathbf{z}_2 \|\right) & \ldots & \varphi\left(\| \mathbf{z}_K - \mathbf{z}_K \|\right) \end{bmatrix} \begin{bmatrix} \delta_1\left(t\right) \\ \delta_2\left(t\right) \\ \vdots \\ \delta_K\left(t\right) \end{bmatrix}$$

which must be solved to obtain the interpolation coefficients $\delta_k\left(t\right)$. Once these weights are found, the value of the function f can be estimated at any set of points $\tilde{\mathbf{Z}} = [\tilde{\mathbf{z}}_1, \ldots, \tilde{\mathbf{z}}_L]'$ with $\tilde{\mathbf{z}}_l \in \mathbb{R}^d$ for $l = 1, \ldots, L$ and time t as

$$f\left(\tilde{\mathbf{Z}}, t\right) \simeq \sum_{k=1}^{K} \delta_k\left(t\right) \varphi\left(\| \tilde{\mathbf{Z}} - \mathbf{z}_k \|\right).$$

Table 1 states the basic function of four RBFs often used in engineering applications, namely the Gaussian RBF, the MQ RBF, the cubic RBF and the TPS RBF (see Koc et al. (2003)).

In the next section we illustrate the application of the RBF interpolation method for approximating put option prices under the CEV and Heston models.

3.2 The CEV Option Prices: Approximation by the RBF Interpolation Method

The solution for American put option prices under the CEV framework is given by the PDE (2) subject to the conditions provided in equations (3a-3d). Following Koc et al.

(2003), we use the Crank-Nicolson averaging to discretize equation (2) in time such that

$$\frac{\mathcal{U}(S,\mathtt{t})-\mathcal{U}(S,\mathtt{t}+\Delta\mathtt{t})}{\Delta\mathtt{t}}+\frac{1}{2}\tilde{\sigma}^2 S^{\beta}\frac{\partial^2\mathcal{U}\left(S,\mathtt{t}+\frac{\Delta\mathtt{t}}{2}\right)}{\partial S^2}+(\mathtt{r}-q)\,S\frac{\partial\mathcal{U}\left(S,\mathtt{t}+\frac{\Delta\mathtt{t}}{2}\right)}{\partial S}-\mathtt{r}\mathcal{U}\left(S,\mathtt{t}+\frac{\Delta\mathtt{t}}{2}\right)=0, \quad (15)$$

where $\Delta\mathtt{t}$ is the time step and $\mathcal{U}\left(S,\mathtt{t}+\frac{\Delta\mathtt{t}}{2}\right)=\frac{1}{2}\left[\mathcal{U}(S,\mathtt{t})+\mathcal{U}(S,\mathtt{t}+\Delta\mathtt{t})\right]$. Next, we separate the variables at time $\mathtt{t}$ and $\mathtt{t}+\Delta\mathtt{t}$ on each side of the equation, so that equation (15) can be rewritten as

$$H_{+}^{CEV}\mathcal{U}(S,\mathtt{t}+\Delta\mathtt{t})=H_{-}^{CEV}\mathcal{U}(S,\mathtt{t}), \quad (16)$$

where H_{+}^{CEV} and H_{-}^{CEV} are the operators

$$H_{+}^{CEV}=1-\frac{\Delta\mathtt{t}}{2}\left[\frac{1}{2}\tilde{\sigma}^2 S^{\beta}\frac{\partial^2}{\partial S^2}+(\mathtt{r}-q)\,S\frac{\partial}{\partial S}-\mathtt{r}\right]$$

$$H_{-}^{CEV}=1+\frac{\Delta\mathtt{t}}{2}\left[\frac{1}{2}\tilde{\sigma}^2 S^{\beta}\frac{\partial^2}{\partial S^2}+(\mathtt{r}-q)\,S\frac{\partial}{\partial S}-\mathtt{r}\right].$$

Now we replace the variable $\mathcal{U}$ in equation (16) by the linear combination of RBFs

$$\mathcal{U}(S,\mathtt{t})\simeq\sum_{k=1}^{K}\delta_k^{CEV}(\mathtt{t})\,\varphi(\| S-S_k \|), \quad k=1,\ldots,K, \quad (17)$$

where the option price $\mathcal{U}$ is evaluated for K values of the stock price (i.e. K centers) such that $S=[S_1,\ldots,S_K]'$; the coefficients $\delta_k^{CEV}(\mathtt{t})$ for $k=1,\ldots,K$ at time $\mathtt{t}$ are the weights and $\varphi(\|\cdot\|)$ is the chosen RBF.

Finally, we obtain the system

$$\sum_{k=1}^{K}\delta_k^{CEV}(\mathtt{t}+\Delta\mathtt{t})\,H_{+}^{CEV}\varphi(\| S-S_k \|)=\sum_{k=1}^{K}\delta_k^{CEV}(\mathtt{t})\,H_{-}^{CEV}\varphi(\| S-S_k \|). \quad (18)$$

To compute the solution $\delta_k^{CEV}(\mathtt{t}+\Delta\mathtt{t})$, we have to solve the linear system (18) iteratively given the values $\delta_k^{CEV}(\mathtt{t})$ from the previous step. The initial value for $\delta_k^{CEV}(\mathtt{t})$ is derived from equation (17) and the initial condition (3a). The boundary conditions must be satisfied through the iterative solution of the system.

The results presented in equation (18) can be easily modified for the pricing of options under the BS model. In this setting, the system of linear equations to be solved is

$$\sum_{k=1}^{K} \delta_k^{BS} (\mathtt{t} + \Delta \mathtt{t}) \, H_+^{BS} \varphi (\| S - S_k \|) = \sum_{k=1}^{K} \delta_k^{BS} (\mathtt{t}) \, H_-^{BS} \varphi (\| S - S_k \|) , \tag{19}$$

where H_+^{BS} and H_-^{BS} are the operators

$$H_+^{BS} = 1 - \frac{\Delta \mathtt{t}}{2} \left[\frac{1}{2} \tilde{\sigma}^2 S^2 \frac{\partial^2}{\partial S^2} + (\mathtt{r} - q) \, S \frac{\partial}{\partial S} - \mathtt{r} \right] ,$$

$$H_-^{BS} = 1 + \frac{\Delta \mathtt{t}}{2} \left[\frac{1}{2} \tilde{\sigma}^2 S^2 \frac{\partial^2}{\partial S^2} + (\mathtt{r} - q) \, S \frac{\partial}{\partial S} - \mathtt{r} \right] .$$

3.3 The Heston Option Prices: Approximation by the RBF Interpolation Method

The solution for American put option prices under the Heston framework is given by the PDE (11) subject to the set of conditions in equations (12a-12f). We follow the same procedure outlined above to approximate the option prices. Following Koc et al. (2003), we use the Crank-Nicolson approach to discretize in time equation (11), yielding

$$\frac{\tilde{U} (\mathcal{X}, V; \mathtt{t}) - \tilde{U} (\mathcal{X}, V; \mathtt{t} + \Delta \mathtt{t})}{\Delta \mathtt{t}} + \frac{1}{2} V \frac{\partial^2 \tilde{U}^{\mathtt{t} + \frac{\Delta \mathtt{t}}{2}}}{\partial \mathcal{X}^2} + \varrho \sigma V \frac{\partial^2 \tilde{U}^{\mathtt{t} + \frac{\Delta \mathtt{t}}{2}}}{\partial \mathcal{X} \partial V} + \frac{1}{2} \sigma^2 V \frac{\partial^2 \tilde{U}^{\mathtt{t} + \frac{\Delta \mathtt{t}}{2}}}{\partial V^2} + \left((\mathtt{r} - q) - \frac{1}{2} V \right) \frac{\partial \tilde{U}^{\mathtt{t} + \frac{\Delta \mathtt{t}}{2}}}{\partial \mathcal{X}} + [\kappa (\theta - V)] \frac{\partial \tilde{U}^{\mathtt{t} + \frac{\Delta \mathtt{t}}{2}}}{\partial V} - \mathtt{r} \tilde{U}^{\mathtt{t} + \frac{\Delta \mathtt{t}}{2}} = 0, \tag{20}$$

where $\Delta \mathtt{t}$ is the time step and $\tilde{U}^{\mathtt{t} + \frac{\Delta \mathtt{t}}{2}} = \frac{1}{2} \left[\tilde{U} (\mathcal{X}, V; \mathtt{t}) + \tilde{U} (\mathcal{X}, V; \mathtt{t} + \Delta \mathtt{t}) \right]$.

The variable $\tilde{U}$ in equation (20) is replaced by the linear combination of RBFs

$$\tilde{U} (\mathcal{Y}; \mathtt{t}) \simeq \sum_{k=1}^{K} \delta_k^{Heston} (\mathtt{t}) \, \varphi (\| \mathcal{Y} - \mathcal{Y}_k \|) , \quad k = 1, \ldots, K, \tag{21}$$

where $\mathcal{Y} = [\mathcal{Y}_1, \ldots, \mathcal{Y}_K]'$ is a two-dimensional vector with K observations (i.e. centers) to be evaluated. Each center is defined as the pair $\mathcal{Y}_k = [\mathcal{X}_k, V_k]$. After separating the elements at time $\mathtt{t}$ and $\mathtt{t} + \Delta \mathtt{t}$ on each side of the equation we obtain the linear system

$$\sum_{k=1}^{K} \delta_k^{Heston} (\mathtt{t} + \Delta \mathtt{t}) \, H_+^{Heston} \varphi (\| \mathcal{Y} - \mathcal{Y}_k \|) = \sum_{k=1}^{K} \delta_k^{Heston} (\mathtt{t}) \, H_-^{Heston} \varphi (\| \mathcal{Y} - \mathcal{Y}_k \|) , \tag{22}$$

where H_{+}^{Heston} and H_{-}^{Heston} are the operators

$$H_{+}^{Heston} = 1 - \frac{\Delta \mathtt{t}}{2} \left[\frac{1}{2}V\frac{\partial^2}{\partial \mathcal{X}^2} + \varrho\sigma V\frac{\partial^2}{\partial \mathcal{X}\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2}{\partial V^2} + \left((\mathtt{r} - q) - \frac{1}{2}V\right)\frac{\partial}{\partial \mathcal{X}} + \left[\kappa\left(\theta - V\right)\right]\frac{\partial}{\partial V} - \mathtt{r}\right]$$

$$H_{-}^{Heston} = 1 + \frac{\Delta \mathtt{t}}{2} \left[\frac{1}{2}V\frac{\partial^2}{\partial \mathcal{X}^2} + \varrho\sigma V\frac{\partial^2}{\partial \mathcal{X}\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2}{\partial V^2} + \left((\mathtt{r} - q) - \frac{1}{2}V\right)\frac{\partial}{\partial \mathcal{X}} + \left[\kappa\left(\theta - V\right)\right]\frac{\partial}{\partial V} - \mathtt{r}\right].$$

The coefficients $\delta_k^{Heston}(\mathtt{t})$ for $k = 1, \ldots, K$ at time $\mathtt{t}$ are the weights and $\varphi(\|\cdot\|)$ is the chosen RBF. To find the weights $\delta_k^{Heston}(\mathtt{t} + \Delta\mathtt{t})$, we have to solve the linear system (22) iteratively given the values $\delta_k^{Heston}(\mathtt{t})$ from the previous step. The initial value for $\delta_k^{Heston}(\mathtt{t})$ is obtained from equation (21) and the final condition (12a). The boundary conditions must be satisfied through the iterative solution of the system.

In general, the option pricing problem is solved by the RBF interpolation method through the iterative solution of a linear system of equations. This solution is based on the operators H_+ and H_- which depend on the assumed model. In addition, the solution of the system must satisfy a set of constraints that are intrinsic to each option type.

4 Numerical Experiments

This section applies the RBF interpolation method to compute the prices of American put options written on a non-dividend-paying stock. We consider two schemes for the volatility, namely the CEV model and the Heston model. The option prices under the BS model are also calculated as a special case of the CEV model with $\beta = 2$. Some results for European options are reported to illustrate the accuracy of the method.

Figure 1 shows the location of the centers employed by the RBF interpolation method in this study. This distribution of points is used in the pricing of options under the CEV model (Panel A) and the Heston model (Panel B). Each plot in this figure contains 200 centers. The Halton sequence is used to place the centers on the areas of interest in the problem. The largest amount of points is added over the space requiring the highest level of accuracy. In the CEV model, the centers are concentrated close to the moneyness $\frac{S}{E}$. In the Heston model, most points are located close to the logarithm of the moneyness $ln\left(\frac{S}{E}\right)$

and the boundary $V = 0$.

Regarding the RBF method, we employ the TPS-RBF stated in equation (28). This particular RBF is chosen for two reasons. First, it does not require the calibration of additional parameters as some RBFs do (e.g. the Gaussian- and MQ-RBF defined in equations (25) and (26), respectively). Second, a previous study in option pricing by Koc et al. (2003) shows the outstanding performance of the TPS-RBF compared with the Cubic-, Gaussian- and MQ-RBF.

The accuracy of the RBF method and its comparison against other numerical techniques is assessed using the following statistical measures: the approximation error and the root-mean-square error (RMSE) (see for example Fasshauer et al. (2004) and Fasshauer (2007)). The distance

$$\left(\hat{O}_h - O_h\right), \tag{23}$$

is defined as the approximation error, where O_h and $\hat{O}_h$ are the h-th observation of the benchmark values vector and the approximated values vector, respectively. The RMSE is computed as

$$RMSE = \sqrt{\frac{1}{\mathcal{H}} \sum_{h=1}^{\mathcal{H}} \left(\hat{O}_h - O_h\right)^2}, \tag{24}$$

where $\mathcal{H}$ is the total number of observations evaluated.

The numerical computations are performed on a DELL machine with a Intel Core 2 DuoE8500 processor, 3.17 GHz CPU speed, 4.00 GB internal memory, 160 GB hard drive disk and operating system Windows 7.

The RBF interpolation results are compared with alternative solutions, including the analytical (or semi-analytical) formula if it is available and two numerical techniques, the FDM and the MCS. In particular, for the CEV model we use a standard FDM with a $N_S \times N_t$ grid, whereas for the Heston model, a two-dimensional problem, we use an operator splitting FDM with a fixed $N_{\mathcal{X}} \times N_V \times N_t$ grid. On the other hand, the MCS method makes use of the antithetic variables and moment matching techniques. For American options, the MCS employes the simulation-based technique introduced in Longstaff and Schwartz (2001). The time is discretized using the Crank-Nicolson averaging.

The option prices, stock prices, strike prices and statistical measures used in this paper are defined in terms of one monetary unit (e.g. one \$ US dollar). In the following, we first discuss the simulation results for the CEV model (Section 4.1), and then for the Heston model (Section 4.2).

4.1 Option Pricing under the CEV Model

In this set of experiments, we consider the pricing of put options under the CEV model. The option parameters are: strike price $E = 10$, time to maturity $\mathtt{t} = 0.25$ years, risk-free interest rate $\mathtt{r} = 10\%$, and volatility $\tilde{\sigma} = 25\%$. We assume the elasticity $\beta = 1$ and hence, a square-root process. The results are presented for a small sample of stock prices $S = [8, 9, 10, 11, 12]$. This sample considers out-of-the-money, at-the-money and in-the-money options.

The option prices are computed in the following way. For American options, the RBF interpolation and the FDM solve the PDE (2) subject to the final and boundary conditions (3a-3d) (i.e. the iterative solution of the system (18) using the RBF method). The solution by the MCS is based on the Euler discretization of the SDE (1). For European contracts, we follow the closed-form solution in Wong and Zhao (2008, pg. 2186, equations (2.2), (2.3) and (2.4)). The pricing of these options by numerical methods requires to run the same procedure employed for American options without the early exercise condition (3d).

The FDM and the RBF interpolation assume the spatial domain $[0, 3E]$ to approximate the semi-infinite domain of the problem $[0, \infty)$. For American options, we use the approximation by the FDM with a 500×500 grid as benchmark. For European options, the benchmark is the closed-form solution.

Table 2 reports the American (Panel A) and European (Panel B) option prices computed by the RBF interpolation method using different numbers of centers. The discretization over time is run with 100 steps. The results are compared with the benchmark. The RMSE and CPU time in seconds are reported. The results show that the RBF method is a robust technique to price options under the CEV model. This method provides highly accurate option prices with a small number of centers that demand low CPU time. For instance, in this experiment the RMSE of the approximation of American option prices fall below 1.4E-4 with just 50 centers and takes 0.12 second. With the same number of centers and similar CPU time, the RMSE for European options is 7.1E-5.

Based on American option prices, Table 3 reports the computed RMSE (Panel A) and CPU time (Panel B) of the approximation for each pair formed by the number of centers over the space and steps for time discretization. The results show the accuracy and efficiency of the RBF interpolation to obtain accurate option prices. For instance, with 100 centers and 200 time steps, the computed option prices achieve a RMSE equal to 9.1E-5 in just 0.24 second.

There exists a trade-off between the level of precision and the efficiency of the method. Lower values of the RMSE can be achieved by increasing the number of centers or the time steps at the cost of higher CPU time. The findings also show that for a very small number of centers, an increase in the amount of time steps does not have a significant effect on the accuracy of the solution, but the CPU time goes up.

Figure 2 plots the RMSE of the approximated American option prices for different values of the elasticity of variance β. The RBF interpolation method is performed with 200 centers and 200 time steps. This figure illustrates the level of accuracy of the numerical method for alternative specifications of the CEV model. We consider values of $\beta \in [0, 3]$. For all values tested, the RMSE is below $7 \times$ 10E-4. The lowest errors are obtained for values of β close or greater than 1.

In the next set of experiments, we compare the results of the RBF interpolation against traditional numerical methods in finance. Table 4 reports the prices of American (Panel A) and European (Panel B) put options computed by the RBF interpolation method, the FDM and the MCS. The benchmark, approximation errors, RMSE and CPU time are reported. The RBF interpolation is performed with 100 centers and 100 time steps while the MCS uses $10,000$ paths. For European options, the approximation by the FDM is carried out with a 400×100 grid. Hence, the number of time steps is comparable for the RBF interpolation and the FDM.

The findings show that the differences in accuracy between the FDM and the RBF interpolation are very small. Nevertheless, both methods achieve higher levels of precision and less time than the MCS technique. For instance, for American options the RBF method yields a RMSE of 1.4E-4 with respect to the FDM. This RMSE is smaller than the same statistic for the MCS technique, 1.8E-3. However, the CPU time used for the latter is considerably higher. On the other hand, with similar CPU time the results in the pricing of European options show that the FDM and the RBF method obtain RMSEs equal to 1.6E-4 and 7.1E-5, respectively.

The MCS method is particularly time-consuming in the pricing of American options due to the Longstaff and Schwartz (2001) algorithm. Moreover, it is important to note that the CPU time reported by the RBF interpolation and the FDM correspond to the solution for the whole domain of the problem. In contrast, the MCS reports a CPU time that considers just the computation of the sample of this experiment. So, this technique would require more time if the whole domain is considered.

The next experiment is an extension to the previous analysis. Figure 3 compares the RMSE and the CPU time of the approximations of the three numerical methods. They are performed using several configurations of the grid size for the FDM, the number of paths for the MCS, and the number of centers and the time steps for the RBF interpolation. Figure 3 confirms the initial conclusion obtained from Table 4. The RBF interpolation and the FDM achieve similar levels of accuracy with low CPU time. Nevertheless, the performance of these techniques depends on the specific configuration. For small grid sizes, the FDM is the fastest method. However, in those cases the mesh-based technique does not achieve the same accuracy as the RBF interpolation. Both methods require similar CPU time to achieve the highest levels of accuracy. On the other hand, the MCS exhibits results with lower accuracy than the RBF interpolation and the FDM, whereas the CPU time to compute the option prices is higher. Moreover, an increase of the number of paths for the MCS only improves marginally the levels of precision in the pricing of American options at the cost of more CPU time.

Special case : The CEV Model with $\beta = 2$.

The BS model is a particular case of the CEV model when the elasticity $\beta = 2$. Table 5 reports the American (Panel A) and European (Panel B) put option prices computed by the RBF interpolation, the FDM and the MCS. The approximation errors, the RMSE and the CPU time are also presented. For American options, the RBF interpolation method and the FDM solve the PDE (6) subject to the final and boundary conditions (3a-3d) (i.e. the iterative solution of the system (19) using the RBF method). The solution by MCS is based on the Euler discretization of the SDE (5). For European options, the closed-form solution is computed following the well-known BS formula given by Black and Scholes (1973, equations (13) and (27) on pages 644 and 647). The pricing of the same contracts by numerical methods requires removing the early exercise condition (3d) of the American option pricing problem.

The conclusions from the general CEV model still hold for this particular case. The RBF interpolation method provides highly accurate option prices with respect to the benchmark for both American and European options. The RBF method and the FDM provide similar levels of accuracy. The MCS is not only the least accurate but also the most time-consuming among the evaluated numerical methods. In particular, this technique becomes very expensive in terms of CPU time for the pricing of American options.

4.2 Option Pricing under the Heston Model

In this set of experiments, we consider the pricing of put options under the Heston model. We specify the following option information: strike price $E = 10$, time to maturity $\mathtt{t} = 0.25$ years and risk-free interest rate $\mathtt{r} = 10\%$. The Heston model assumes that the variance V follows a stochastic process with mean reversion speed $\kappa = 5$, long-run mean $\theta = 0.16$, volatility of the volatility $\sigma = 0.9$. The correlation $\varrho = 0.1$ and the market price of risk $\zeta = 0$. The option prices are computed for the stock price $S = [8, 9, 10, 11, 12]$ and initial value of the variance $V_0 = 6.25\%$ and $V_0 = 25\%$. This configuration has been used in the literature on the pricing of American options by Oosterlee (2003), Ikonen and Toivanen (2004), Ikonen and Toivanen (2007), Ikonen and Toivanen (2008) and Persson and Sydow (2010). We keep the same specification to compare the RBF interpolation results with those computed in those studies.

Option prices are computed in the following way. For American contracts, the RBF interpolation and the FDM solve the PDE (11) subject to the final and boundary conditions in equations (12a-12f) (i.e. the iterative solution of the system (22) using the RBF method). The solution by the MCS is based on the Euler discretization of the SDE (7-9). The Heston model does not have a closed-form solution. Nevertheless, we use the semi-analytical solution provided by Heston (1993, equations (17) and (18) on page 331) for European call options and use the put-call parity condition to obtain the put option prices. The pricing of European options by numerical methods implies performing the same routine described for American contracts without considering the early condition (12f).

The FDM and the RBF interpolation assume the spatial domain $[S_{min}, S_{max}] \times [V_{min}, V_{max}]$ as $[0, 3E] \times [0, 1]$ to approximate the semi-infinite domain of the problem $[0, \infty) \times [0, \infty)$. We consider the solutions given by Ikonen and Toivanen (2008), using the componentwise splitting method with a $5120 \times 2048 \times 2050$ grid, and Heston (1993) with a semi-analytical formula as the benchmark for American and European options, respectively.

Table 6 reports the American (Panel A) and European (Panel B) option prices computed by the RBF interpolation using different number of centers. The discretization over time is implemented with 100 steps. The approximated option prices are compared with the benchmark. The RMSE and the CPU time of the approach are reported. Similar to the findings in Section 4.1 for the CEV model, Table 6 shows that the RBF interpolation is an useful technique in the pricing of options under the Heston model. The method achieves very accurate, reliable and efficient results. For example, with 100 centers and 100 time steps the RBF interpolation yields American option prices with a RMSE of 5.2E-3 in just 0.094 second. In addition, the accuracy of the approximation improves as the number of points increases at the cost of higher CPU time. For instance, with 300 points and the same number of time steps, the RMSE falls below 6.04E-4 but the time goes up quickly to 0.92 second. Similar conclusions can be drawn from the approximation of the European option prices.

Table 7 reports the RMSE (Panel A) and the CPU time (Panel B) of the approximation for each pair formed by an increasing number of centers over the space and steps for time discretization. The results confirm the robustness and precision of the RBF interpolation. The method is able to compute option prices with small RMSE and demands low CPU time. For instance, with just 200 centers and 50 time steps, the RMSE of the approximation is 1.2E-3 and takes 0.20 second. Using 100 time steps, the RMSE falls to 9.9E-4. It is clear that higher levels of accuracy are achieved with larger values for the number of points and time steps. However, increasing time steps could lead to longer CPU time without a significant improvement in accuracy. For example, the RMSE of approximation with 150 centers and 100 time steps is 2.0E-3 and takes around 0.2 second. With the same number of centers and 400 time steps the RMSE falls to1.9E-3 but the CPU time increases almost 4 times.

It is important to note that the level of accuracy for the RBF method depends in great part on the distribution of centers. The precision of the approximation can also be enhanced using a more efficient distribution of points on the areas of most interest.

In the next experiment, we compare the results of the RBF interpolation against those computed by the operator splitting FDM and the MCS method. Table 8 reports both American (Panel A) and European (Panel B) option prices under the Heston model. This table includes the approximation errors, the RMSE of the option prices and the CPU time. The RBF method uses 200 centers. The time is discretized with 100 steps. The operator splitting FDM is performed with a $160 \times 64 \times 64$ grid. This grid size is considered in Ikonen and Toivanen (2004) to deal with the Heston model using the same FDM. The MCS is carried out with $10,000$ paths.

The results show that the RBF interpolation method achieves higher levels of accuracy and lower CPU time than the FDM and the MCS. For instance, for American options, the approximation by the RBF method reports a RMSE of 9.9E-4 while the statistic for the MCS is 7.9E-3. However, the latter requires a CPU time considerably higher than the one needed by the meshfree method. Similar results are obtained for European options. Both the FDM and the MCS provide results with less accuracy and longer CPU time than the RBF interpolation.

In order to extend the previous analysis, the next experiment compares the efficiency of the methods. Figure 4 scatters the RMSE of the approximation against the CPU time required to compute the option prices for both American (Panel A) and European (Panel B) contracts. The comparison is carried out using different grid sizes for the operator splitting FDM, the number of paths for the MCS and the number of centers and time steps for the RBF interpolation method.

The results show that the RBF method achieves more accurate and clearly faster approximations for option prices than the operator splitting FDM and the traditional MCS. In this two-dimensional problem, the FDM is no longer the faster method. On the contrary, this approach requires finer grids to achieve similar levels of accuracy compared with the RBF method, becoming a time-consuming approach. The high CPU time required by the operator splitting FDM is consistent with the findings in Ikonen and Toivanen (2004). The MCS method provides the least accurate results. This technique is also time-consuming. Therefore, the RBF interpolation proves to be superior in this application to the other two methods.

Finally, Table 9 compares the approximated American option prices by the RBF interpolation with the results of recent studies in the literature, including Zvan et al. (1998), Oosterlee (2003), Ikonen and Toivanen (2004), Ikonen and Toivanen (2008) and Persson and Sydow (2010). These references use novel techniques based on mesh-based methods and the MCS. The option prices are compared in terms of the RMSE. This statistic is computed assuming that the benchmark is the solution by the RBF interpolation with 1000 centers and 100 time steps. The prices and the RMSE are presented for values of the variance $V_0 = 6.25\%$ (Panel A) and $V_0 = 25\%$ (Panel B). The results show that the RBF method computes option prices very close to those provided in the references. In fact, the differences between the reported approaches are very small. In particular, the RBF interpolation results are in good agreement with the option prices reported in Ikonen and Toivanen (2004) and Ikonen and Toivanen (2008). The findings offer evidence of the accuracy of the RBF interpolation method to solve the option pricing problem under the Heston model.

5 Conclusion

In this paper, we propose the RBF interpolation method for pricing American options. In particular, we consider two processes for the volatility, the CEV model and the Heston model. They correspond to processes for modelling the local and stochastic volatility, respectively. The CEV model is performed with a parameter $\beta = 1$, hence a square-root process. The BS prices are calculated as a special case of the CEV model with $\beta = 2$. In general, the option prices are computed as the solution of PDEs subject to a set of conditions associated with the features of the option. Additional results for European options are also reported.

The results achieved by the RBF interpolation method are compared with those computed by alternative solutions, namely the analytical formula if available, the FDM and the MCS method. The option prices are also compared with results in recent studies. For the CEV model, a one-dimensional problem, the RBF method is superior in accuracy and efficiency to the MCS, and shows similar performance to the FDM. The gains of using the meshfree method are evident in the Heston model. In this two-dimensional problem, the results show that the RBF method achieves more accurate and faster option prices than both the operator splitting FDM and the traditional MCS.

There exists a trade-off between the levels of precision and the computational efficiency of the method. The accuracy of the approximation improves as the number of centers increases at the cost of higher CPU time. Moreover, the level of accuracy in the RBF interpolation depends critically on the distribution of centers. Higher levels of precision can be obtained using a more efficient distribution of points on the areas of most interest.

In general, the findings show that the RBF interpolation is a robust, reliable and effective technique to price options under the CEV and Heston models. The method requires few time steps to obtain solutions converging to accurate option prices. The RBF method provides highly accurate and efficient approximations of American option prices, often outperforming the chosen benchmark, the other two numerical methods and the recent results in the literature.

References

AitSahlia, F., Goswami, M., Guha, S., 2010. American option pricing under stochastic volatility: An empirical evaluation. Computational Management Science 7 (2), 189–206.

Bakshi, G., Cao, C., Chen, Z., 1997. Empirical performance of alternative option pricing models. Journal of Finance 52 (5), 2003–2049.

Beliaeva, N. A., Nawalkha, S., 2010. A simple approach to pricing American options under the Heston stochastic volatility model. Journal of Derivatives 17 (4), 25–43.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81 (3), 637–654.

Boyle, P., Kolkiewicz, A., Tan, K., 2003. Improved simulation method for pricing high-dimensional American derivatives. Mathematics and Computers in Simulation 62 (3), 315–322.

Boyle, P., Tian, Y., 1999. Pricing lookback and barrier options under the CEV process. Journal of Financial and Quantitative Analysis 34 (2), 241–264.

Chang, C.-C., Chung, S.-L., Stapleton, R. C., 2007. Richardson extrapolation techniques for the pricing of american-style options. Journal of Futures Markets 27 (8), 791–817.

Chen, H.-C., Chen, D. M., Chung, S.-L., 2002. The accuracy and efficiency of alternative option pricing approaches relative to a log-transformed trinomial model. Journal of Futures Markets 22 (6), 557–577.

Chen, R., Lee, C., Lee, H., 2009. Empirical performance of the constant elasticity variance option pricing model. Review of Pacific Basin Financial Markets and Policies 12 (2), 177–217.

Choi, S., Fouque, J., Kim, J., 2010. Option pricing under hybrid stochastic and local volatility. Working Paper, PSTAT University of California, Santa Barbara.

Cont, R., 2001. Empirical properties of asset returns: Stylized facts and statistical issues. Quantitative Finance 1 (2), 223–236.

Cox, J., Ingersoll, J., Ross, S., 1985. A theory of the term structure of interest rates. Econometrica 53 (2), 385–407.

Cox, J., Ross, S., 1976. The valuation of options for alternative stochastic processes. Journal of Financial Economics 3 (1), 145–166.

Daum, F., 2005. Nonlinear filters: Beyond the Kalman filter. Aerospace and Electronic Systems Magazine, IEEE 20 (8), 57–69.

Daum, F., Krichman, M., 2006. Meshfree adjoint methods for nonlinear filtering. In: Aerospace Conference, 2006. IEEE, Big Sky, MT, pp. 1–16.

Dell'Era, M., 2010. Vanilla option pricing on stochastic volatility market models. International Review of Applied Financial Issues and Economics 3 (4), in Press.

Duck, P. W., Newton, D. P., Widdicks, M., Leung, Y., 2005. Enhancing the accruacy of pricing American and Bermudan options. Journal of Derivatives 12 (4), 34–44.

Duffy, D., 2006. Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach. John Wiley & Sons, Chichester.

Düring, B., Fournié, M., 2010. Compact finite difference scheme for option pricing in Heston's model. In: Simos, T., Psihoyios, G., Tsitouras, C. (Eds.), AIP Conference Proceedings: Numerical Analysis and Applied Mathematics. No. 1. American Institute of Physics, Melvill, NY, pp. 219–222.

Dutt, S. L., Welke, G. M., 2008. Just-in-time Monte Carlo for path-dependent American options. Journal of Derivatives 15 (4), 29–47.

Fasshauer, G., 2006. Meshfree methods. In: Rieth, M., Schommers, W. (Eds.), Handbook of Theoretical and Computational Nanotechnology. American Scientific Publishers, pp. 33–97.

Fasshauer, G., 2007. Meshfree Approximation Methods with MATLAB. World Scientific Publishers, Singapore.

Fasshauer, G., Khaliq, A., Voss, D., 2004. Using meshfree approximation for multi-asset American options. Journal of Chinese Institute of Engineers 27 (4), 563–571.

Fasshauer, G., Khaliq, A., Voss, D., 2008. A parallel time stepping approach using meshfree approximations for pricing options with non-smooth payoffs. Journal of Risk 10 (4), 135–142.

Fatone, L., Marioni, F., Recchioni, M., Zirilli, F., 2009. An explicitly solvable multi-scale stochastic volatility model: Option pricing and calibration problems. Journal of Futures Markets 29 (9), 862–893.

Fouque, J., Papanicolaou, G., Sircar, K., 2000. Derivatives in Financial Markets with Stochastic Volatility. Cambridge University Press, Cambridge.

Goto, Y., Fei, Z., Kan, S., Kita, E., 2007. Options valuation by using radial basis function approximation. Engineering Analysis with Boundary Elements 31 (10), 836 – 843.

Heston, S., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies 6 (2), 327–343.

Hon, Y., 2002. A quasi-radial basis functions method for American option pricing. Computers & Mathematics with Applications 43 (3), 513–524.

Hon, Y., Mao, X., 1999. A radial basis function method for solving options pricing model. Financial Engineering 8 (1), 31–49.

Ikonen, S., Toivanen, J., 2004. Operator splitting methods for American options with stochastic volatility. In: Neittanmaki, P., Rossi, T., Korotov, S., Oñate, E., Periaux, J. (Eds.), European Congress on Computational Methods on Applied Sciences and Engineering. Jyväskylä, pp. 1–19.

Ikonen, S., Toivanen, J., 2007. Componentwise splitting methods for pricing American options under stochastic volatility. International Journal of Theoretical and Applied Finance 10 (2), 331–361.

Ikonen, S., Toivanen, J., 2008. Efficient numerical methods for pricing American options under stochastic volatility. Numerical Methods for Partial Differential Equations 24 (1), 104–126.

Iske, A., 2004. Multiresolution Methods in Scattered Data Modelling. Lecture Notes in Computational Science and Engineering. Springer, Heidelberg.

Kelly, M., 2009. Evaluation of financial options using radial basis functions in Mathematica. Mathematica Journal 11 (3), 333–357.

Koc, M., Boztosun, I., Boztosun, D., 2003. On the numerical solution of Black-Scholes equation. In: International Workshop on Meshfree Methods. Instituto Superior Técnico - ICIST, Lisbon, Portugal, pp. 1–6.

Larsson, E., Ahlander, K., Hall, A., 2008. Multi-dimensional option pricing using radial basis functions and the generalized Fourier transform. Journal of Computational and Applied Mathematics 222 (1), 175–192.

Li, S., Liu, W., 2004. Meshfree Particle Methods. Springer, Berlin.

Liu, G., 2003. Mesh Free Methods: Moving Beyond the Finite Element Method. CRC Press, London.

Liu, G., Gu, Y., 2005. An Introduction to Meshfree Methods and their Programming. Springer, Dordrecht.

Liu, G., Liu, M., 2003. Smoothed Particle Hydrodynamics: A Meshfree Particle Method. World Scientific Publishing, Singapore.

Liu, Q., 2010. Pricing american options by canonical least-squares monte carlo. Journal of Futures Markets 30 (2), 175–187.

Longstaff, F., Schwartz, E., 2001. Valuing American options by simulation: A simple least-squares approach. Review of Financial Studies 14 (1), 113–147.

Mei, L., Cheng, P., 2008. Multivariable option pricing using quasi-interpolation based on radial basis functions. In: 4th International Conference on Intelligent Computing. SPIE, Shangai, China, pp. 620–627.

Merton, R., 1973. Theory of rational option pricing. Bell Journal of Economics and Management Science 4 (1), 141–183.

Mitra, S., 2010. Regime switching stochastic volatility option pricing. International Journal of Finance Markets and Derivatives 1 (2), 213–242.

Oosterlee, C., 2003. On multigrid for linear complementary problems with application to American-style options. Electronic Transactions on Numerical Analysis 15 (1), 165–185.

O'Sullivan, C., O'Sullivan, S., 2010. Pricing option under Heston's stochastic volatility model via accelerated explicit finite differencing methods. Working Paper 201031, UCD Geary Institute, University College Dublin.

Park, S. H., Kim, J. H., 2011. Asymptotic option pricing under the CEV difussion. Journal of Mathematics Analysis and Applications 375 (2), 490–501.

Persson, J., Sydow, L. V., 2010. Pricing American options using a space-time adaptive finite difference method. Mathematics and Computers in Simulation 80 (9), 1922–1935.

Pettersson, U., Larsson, E., Marcusson, G., Persson, J., 2008. Improved radial basis function methods for multi-dimensional option pricing. Journal of Computational and Applied Mathematics 222 (1), 82–93.

Rubinstein, M., 1994. Implied binomial trees. Journal of Finance 49 (3), 771–818.

Vidal, J., 2009. Pricing American options under the constant elasticity of variance model and subject to bankruptcy. Journal of Financial and Quantitative Analysis 44 (5), 1231–1263.

Widdicks, M., Andricopoulos, A. D., Newton, D. P., Duck, P. W., 2002. On the enhanced convergence of standard lattice methods for option pricing. Journal of Futures Markets 22 (4), 315–338.

Wong, H., Zhao, J., 2008. An artificial boundary method for American option pricing under the CEV model. SIAM Journal on Numerical Analysis 46 (4), 2183–2209.

Zhang, J., Shu, J., 2003. Pricing S&P 500 index options with Heston's model. In: Computational Intelligence for Financial Engineering, 2003. IEEE International Conference. IEEE Xplore, Hong Kong, pp. 85–92.

Zhang, Z., Lim, K.-G., 2006. A non-lattice pricing model of american options under stochastic volatility. Journal of Futures Markets 26 (5), 417–448.

Zvan, R., Forsyth, P., Vetzal, K., 1998. Penalty methods for American options with stochastic volatility. Journal of Computational and Applied Mathematics 91 (2), 199–218.

Table 1. Some Examples of Popular RBFs

$$\text{Gaussian RBF: } \varphi(c_k) = \exp^{-\varepsilon^2 c_k^2} \tag{25}$$

$$\text{MQ-RBF: } \varphi(c_k) = \sqrt{\varepsilon^2 + c_k^2} \tag{26}$$

$$\text{Cubic RBF: } \varphi(c_k) = c_k^3 \tag{27}$$

$$\text{TPS-RBF: } \varphi(c_k) = c_k^4 \ln(c_k) \tag{28}$$

where $c_k = \parallel \mathbf{Z} - \mathbf{z}_k \parallel$.

$\varphi(c_k)$ is the basic function of the RBF $\mathbf{\Phi}_k$ centered on c_k. The latter is defined as $c_k = \parallel \mathbf{Z} - \mathbf{z}_k \parallel$ where $\parallel \cdot \parallel$ is the Euclidean norm, $\mathbf{Z}$ is the set of centers $[\mathbf{z}_1, \ldots, \mathbf{z}_K]^{'}$, and $\mathbf{z}_k \in \mathbb{R}^d$ is the k-th center. The constant ε is a shape parameter.

Table 2. CEV Option Prices by the RBF Interpolation Method

(A) American Options

Stock	FDM	RBF Interpolation (Number of Centers)					
Price	Benchmark	30	50	80	100	150	200
8	2.0000	1.9999	2.0000	2.0000	2.0000	2.0000	2.0000
9	1.0000	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000
10	0.0864	0.0771	0.0861	0.0862	0.0862	0.0862	0.0862
11	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
RMSE		4.2E-03	1.4E-04	1.1E-04	1.1E-04	9.2E-05	7.9E-05
CPU Time	1.46	0.03	0.12	0.47	1.67	3.46	7.21

(B) European Options

Stock	Analytical	RBF Interpolation (Number of Centers)					
Price	Solution	30	50	80	100	150	200
8	1.7531	1.7532	1.7531	1.7531	1.7531	1.7531	1.7531
9	0.7565	0.7564	0.7565	0.7565	0.7565	0.7565	0.7565
10	0.0624	0.0660	0.0626	0.0626	0.0624	0.0624	0.0625
11	0.0001	0.0001	0.0000	0.0001	0.0001	0.0001	0.0001
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
RMSE		1.6E-03	7.1E-05	6.5E-05	1.6E-05	1.3E-05	1.4E-05
CPU Time		0.04	0.13	0.55	1.57	3.60	7.37

This table reports the American (Panel A) and European (Panel B) put option prices under the CEV model for a sample of stock values $S = [8, 9, 10, 11, 12]$. The option parameters are: $E = 10$, $\mathtt{t} = 0.25$ years, $\mathtt{r} = 10\%$, $\tilde{\sigma} = 25\%$ and $\beta = 1$. The option prices are computed by the RBF interpolation method for several configurations of the number of centers. The discretization over time considers 100 steps. The table also reports the RMSE and the CPU time of the solution. The benchmark for American options is the approximation by the FDM with a 500×500 grid. In the case of European options, the benchmark is the analytical solution given by Wong and Zhao (2008, p. 2186, equations (2.2), (2.3) and (2.4)). The stock prices, option prices and the RMSE are defined in terms of a monetary unit (e.g. $ US dollar). The CPU time is reported in seconds.

Table 3. CEV Option Prices: Accuracy and Efficiency Analysis

(A) RMSE

Time	RBF Interpolation (Number of Centers)						
Steps	30	50	100	200	500	700	1000
20	1.3E-02	4.9E-03	7.0E-04	5.4E-04	5.1E-04	6.5E-04	6.4E-04
50	1.3E-02	4.4E-03	2.8E-04	2.4E-04	2.0E-04	1.9E-04	2.1E-04
100	1.3E-02	4.2E-03	1.4E-04	1.1E-04	9.2E-05	7.9E-05	1.1E-04
200	1.3E-02	4.1E-03	9.1E-05	5.8E-05	5.0E-05	4.1E-05	5.4E-05
300	1.3E-02	4.1E-03	7.6E-05	5.8E-05	4.7E-05	3.9E-05	4.6E-05
400	1.3E-02	4.1E-03	7.2E-05	5.5E-05	5.6E-05	4.7E-05	3.9E-05
500	1.3E-02	4.1E-03	6.5E-05	5.1E-05	5.6E-05	5.5E-05	4.5E-05

(B) CPU Time

Time	RBF Interpolation (Number of Centers)						
Steps	30	50	100	200	500	700	1000
20	0.00	0.01	0.03	0.10	0.75	1.66	4.19
50	0.01	0.02	0.07	0.25	1.72	3.75	10.18
100	0.02	0.04	0.12	0.49	3.67	7.19	19.60
200	0.04	0.09	0.24	0.91	7.30	14.31	38.20
300	0.06	0.13	0.36	1.22	10.97	21.68	54.76
400	0.09	0.17	0.47	1.63	14.15	28.74	75.61
500	0.10	0.21	0.54	2.19	17.86	35.88	96.11

This table reports the RMSE (Panel A) and the CPU time (Panel B) in the approximation of American put options prices under the CEV model. The approximation is performed by the RBF interpolation method for several configurations of the number of centers and time steps. The option parameters are: $E = 10$, $\mathtt{t} = 0.25$ years, $\mathtt{r} = 10\%$, $\tilde{\sigma} = 25\%$ and $\beta = 1$. The RMSE is computed with the option prices that correspond to stock values $S = [8, 9, 10, 11, 12]$. The benchmark for American options is the approximation by the FDM with a 500×500 grid. The RMSE is defined in terms of a monetary unit (e.g. \$ US dollar). The CPU time is reported in seconds.

Table 4. CEV Option Prices: A Comparison of Numerical Methods

(A) American Options

Stock Price	Method			Approximation Error	
	FDM Benchmark	MC	RBF	MC	RBF
8.0	2.0000	1.9972	2.0000	-2.8E-03	3.8E-06
9.0	1.0000	0.9972	1.0000	-2.8E-03	4.1E-06
10.0	0.0864	0.0861	0.0861	-2.9E-04	-3.1E-04
11.0	0.0001	0.0002	0.0001	5.6E-05	-5.4E-05
12.0	0.0000	0.0000	0.0000	-2.2E-09	-7.1E-10
RMSE				1.8E-03	1.4E-04
CPU Time	1.36			1.67	0.11

(B) European Options

Stock Price	Analytical Solution	Method			Approximation Error		
		FDM	MC	RBF	FDM	MC	RBF
8.0	1.7531	1.7531	1.7531	1.7531	1.7E-07	4.9E-05	3.6E-05
9.0	0.7565	0.7568	0.7566	0.7565	2.7E-04	1.2E-04	1.5E-05
10.0	0.0624	0.0627	0.0634	0.0626	2.2E-04	9.7E-04	1.5E-04
11.0	0.0001	0.0001	0.0001	0.0000	7.4E-06	2.2E-05	-5.2E-05
12.0	0.0000	0.0000	0.0000	0.0000	1.1E-09	-1.4E-09	9.2E-09
RMSE					1.6E-04	4.4E-04	7.1E-05
CPU Time					0.13	0.41	0.10

This table reports the prices, the approximation errors, the RMSE and the CPU time of the pricing of American (Panel A) and European (Panel B) put options under the CEV model. The option prices are computed by the FDM, the MCS and the RBF interpolation. The option parameters are: $E = 10$, $\mathtt{t} = 0.25$ years, $\mathtt{r} = 10\%$, $\tilde{\sigma} = 25\%$ and $\beta = 1$. The benchmark for American options is the approximation by the FDM with a 500×500 grid. For European options the benchmark is the analytical solution in Wong and Zhao (2008, p. 2186, equations (2.2), (2.3) and (2.4)). The RBF interpolation is performed with 100 centers and 100 time steps while the MCS uses $10,000$ paths. For European options, the approximation by the FDM is carried out with a 400×100 grid. The option prices, the approximation errors and the RMSE are defined in terms of a monetary unit (e.g. $ US dollar). The CPU time is reported in seconds.

Table 5. BS Option Prices: A Comparison of Numerical Methods

(A) American Options

Stock Price	Numerical Methods: FDM Benchmark	MC	RBF	Approximation Error: MC	RBF
8.0	2.0000	1.9975	2.0000	-2.5E-03	4.4E-06
9.0	1.0303	1.0255	1.0300	-4.7E-03	-3.1E-04
10.0	0.4024	0.4068	0.4021	4.3E-03	-3.3E-04
11.0	0.1207	0.1227	0.1205	2.0E-03	-1.8E-04
12.0	0.0282	0.0305	0.0281	2.3E-03	-6.8E-05
RMSE				3.4E-03	2.2E-04
CPU Time	1.30			1.69	0.10

(B) European Options

Stock Price	Analytical Solution	Numerical Methods: FDM	MC	RBF	Approximation Error: FDM	MC	RBF
8.0	1.7796	1.7798	1.7801	1.7795	1.4E-04	4.8E-04	-9.4E-05
9.0	0.9370	0.9370	0.9376	0.9370	1.9E-05	5.9E-04	-1.5E-05
10.0	0.3785	0.3787	0.3841	0.3785	1.6E-04	5.5E-03	-2.4E-05
11.0	0.1157	0.1158	0.1161	0.1156	1.1E-04	4.6E-04	-3.8E-05
12.0	0.0273	0.0273	0.0286	0.0273	2.7E-05	1.3E-03	-3.0E-05
RMSE					1.1E-04	2.6E-03	4.9E-05
CPU Time					0.14	0.27	0.10

This table reports the prices and the approximation errors of American (Panel A) and European (Panel B) put options under the BS model. The option prices are computed by the FDM, the MCS and the RBF interpolation. The option parameters are: $E = 10$, $\mathfrak{t} = 0.25$ years, $\mathbf{r} = 10\%$, $\tilde{\sigma} = 25\%$ and $\beta = 2$. The benchmark for American options is the approximation by the FDM with a 500×500 grid. For European options the benchmark is the analytical solution given by Black and Scholes (1973, p. 644 and 647, equations (13) and (27)). The RBF interpolation is performed with 100 centers and 100 time steps while the MCS uses $10,000$ paths. For European options, the approximation by the FDM is carried out with a 400×100 grid. The option prices, the approximation errors and the RMSE are defined in terms of a monetary unit ($ US dollar). The CPU time is reported in seconds.

Table 6. Heston Option Prices by the RBF Interpolation Method

(A) American Options

Stock Price	Var. %	Benchmark FDM	RBF Interpolation (Number of Centers) 50	100	200	300	500	1000
8.0	6.25	2.0000	2.0025	2.0056	1.9983	1.9992	1.9986	2.0000
9.0	6.25	1.1076	1.0954	1.1055	1.1063	1.1065	1.1065	1.1070
10.0	6.25	0.5200	0.4985	0.5204	0.5205	0.5195	0.5196	0.5199
11.0	6.25	0.2137	0.1810	0.2112	0.2140	0.2134	0.2138	0.2137
12.0	6.25	0.0820	0.0580	0.0838	0.0811	0.0819	0.0820	0.0819
8.0	25	2.0784	2.0388	2.0697	2.0770	2.0778	2.0781	2.0780
9.0	25	1.3336	1.2974	1.3307	1.3325	1.3330	1.3331	1.3332
10.0	25	0.7960	0.7590	0.7900	0.7953	0.7954	0.7957	0.7957
11.0	25	0.4483	0.4119	0.4412	0.4480	0.4478	0.4483	0.4481
12.0	25	0.2428	0.2132	0.2357	0.2425	0.2426	0.2431	0.2427
RMSE			3.0E-02	5.2E-03	9.9E-04	6.0E-04	6.3E-04	2.9E-04
CPU Time			0.031	0.094	0.359	0.921	2.870	14.819

(B) European Options

Stock Price	Var. %	Benchmark SA. Sol	RBF Interpolation (Number of Centers) 50	100	200	300	500	1000
8.0	6.25	1.8389	1.8439	1.8449	1.8411	1.8382	1.8385	1.8388
9.0	6.25	1.0483	1.0247	1.0470	1.0494	1.0485	1.0486	1.0485
10.0	6.25	0.5015	0.4746	0.5010	0.5026	0.5014	0.5019	0.5017
11.0	6.25	0.2082	0.1737	0.2051	0.2086	0.2081	0.2085	0.2083
12.0	6.25	0.0804	0.0559	0.0814	0.0795	0.0804	0.0804	0.0803
8.0	25	1.9773	1.9072	1.9724	1.9768	1.9768	1.9774	1.9773
9.0	25	1.2800	1.2350	1.2751	1.2800	1.2798	1.2803	1.2800
10.0	25	0.7697	0.7300	0.7630	0.7699	0.7696	0.7701	0.7698
11.0	25	0.4360	0.3983	0.4286	0.4362	0.4360	0.4365	0.4361
12.0	25	0.2373	0.2065	0.2298	0.2371	0.2373	0.2378	0.2373
RMSE			3.7E-02	5.0E-03	9.4E-04	2.8E-04	3.5E-04	1.1E-04
CPU Time			0.031	0.109	0.406	1.029	2.932	14.944

This table reports the American (Panel A) and European (Panel B) put option prices under the Heston model. The option parameters are: $E = 10$, $\mathtt{t} = 0.25$ years and $\mathtt{r} = 10\%$. The Heston model assumes that $\kappa = 5$, $\theta = 0.16$, $\sigma = 0.9$. and $\varrho = 0.1$. The market price of risk $\zeta = 0$. The option prices are computed for stock prices $S = [8, 9, 10, 11, 12]$ and initial values of the variance $V_0 = 6.25\%$ and $V_0 = 25\%$. The option prices are approximated by the RBF interpolation method for several configurations of the number of centers. The benchmark for American options is given by Ikonen and Toivanen (2008), using the componentwise splitting method with a $5120 \times 2048 \times 2050$ grid. In the case of European options, the benchmark is computed with the semi-analytical formula given in Heston (1993, equations (17) and (18) on page 331) plus the put-call parity condition. The discretization over time considers 100 steps. The option prices and the RMSE are defined in terms of a monetary unit ($ US dollar). The CPU time is reported in seconds.

Table 7. Heston Option Prices: Accuracy and Efficiency Analysis

(A) RMSE

Time	RBF Interpolation (Number of Centers)						
Steps	80	100	150	200	300	600	1000
10	9.1E-03	6.9E-03	3.8E-03	3.0E-03	3.6E-03	4.1E-03	5.3E-03
20	8.2E-03	6.0E-03	2.9E-03	2.1E-03	1.6E-03	1.6E-03	1.8E-03
40	7.5E-03	5.4E-03	2.3E-03	1.3E-03	1.0E-03	1.0E-03	7.6E-04
50	7.4E-03	5.3E-03	2.2E-03	1.2E-03	8.7E-04	8.6E-04	6.0E-04
100	7.2E-03	5.2E-03	2.0E-03	9.9E-04	6.0E-04	5.6E-04	2.9E-04
200	7.0E-03	5.1E-03	2.0E-03	9.2E-04	5.0E-04	4.7E-04	1.8E-04
300	7.0E-03	5.1E-03	1.9E-03	9.0E-04	4.8E-04	4.5E-04	1.8E-04
400	7.0E-03	5.0E-03	1.9E-03	8.9E-04	4.6E-04	4.5E-04	1.8E-04

(B) CPU Time

Time	RBF Interpolation (Number of Centers)						
Steps	80	100	150	200	300	600	1000
10	0.02	0.02	0.03	0.06	0.16	0.77	2.34
20	0.02	0.03	0.06	0.09	0.25	1.12	3.79
40	0.03	0.03	0.09	0.17	0.41	1.97	6.66
50	0.03	0.06	0.11	0.20	0.48	2.34	8.15
100	0.06	0.08	0.20	0.34	0.91	4.29	15.43
200	0.13	0.17	0.39	0.66	1.73	8.46	29.87
300	0.19	0.26	0.58	0.98	2.56	12.32	44.46
400	0.25	0.36	0.78	1.31	3.39	16.41	58.72

This table reports the RMSE (Panel A) and the CPU time (Panel B) of the approximation of American put options prices under the Heston model. The approach is performed by the RBF interpolation method for several configurations of the number of centers and time steps. The option parameters are: $E = 10$, $\mathtt{t} = 0.25$ years and $\mathtt{r} = 10\%$. The Heston model assumes that $\kappa = 5$, $\theta = 0.16$, $\sigma = 0.9$ and $\varrho = 0.1$. The market price of risk $\zeta = 0$. The option prices are computed for stock values $S = [8, 9, 10, 11, 12]$ and initial values of the variance $V_0 = 6.25\%$ and $V_0 = 25\%$. The benchmark for American options is given by Ikonen and Toivanen (2008), using the componentwise splitting method with a $5120 \times 2048 \times 2050$ grid. The RMSE is defined in terms of a monetary unit (e.g. \$ US dollar). The CPU time is reported in seconds.

Table 8. Heston Option Prices: A Comparison of Numerical Methods

(A) American Options

Stock Price	Var. %	Numerical Methods: FDM Benchmark	Numerical Methods: MC	Numerical Methods: RBF	Approximation Error: MC	Approximation Error: RBF
8.0	6.25	2.0000	1.9977	1.9983	-2.3E-03	-1.7E-03
9.0	6.25	1.1076	1.0914	1.1063	-1.6E-02	-1.3E-03
10.0	6.25	0.5200	0.5115	0.5205	-8.5E-03	5.2E-04
11.0	6.25	0.2137	0.2124	0.2140	-1.3E-03	3.4E-04
12.0	6.25	0.0820	0.0820	0.0811	-1.4E-05	-9.7E-04
8.0	25	2.0784	2.0742	2.0770	-4.2E-03	-1.4E-03
9.0	25	1.3336	1.3216	1.3325	-1.2E-02	-1.1E-03
10.0	25	0.7960	0.7852	0.7953	-1.1E-02	-6.5E-04
11.0	25	0.4483	0.4457	0.4480	-2.5E-03	-2.6E-04
12.0	25	0.2428	0.2427	0.2425	-1.0E-04	-3.1E-04
RMSE					7.9E-03	9.9E-04
CPU					6.50	0.57

(B) European Options

Stock Price	Var. %	Analytical Solution	Numerical Methods: FDM	Numerical Methods: MC	Numerical Methods: RBF	Approximation Error: FDM	Approximation Error: MC	Approximation Error: RBF
8.0	6.25	1.8389	1.8357	1.8373	1.8411	-3.2E-03	-1.5E-03	2.3E-03
9.0	6.25	1.0483	1.0476	1.0478	1.0494	-7.6E-04	-5.4E-04	1.0E-03
10.0	6.25	0.5015	0.5019	0.5001	0.5026	4.7E-04	-1.4E-03	1.2E-03
11.0	6.25	0.2082	0.2097	0.2079	0.2086	1.5E-03	-3.4E-04	4.6E-04
12.0	6.25	0.0804	0.0825	0.0807	0.0795	2.1E-03	2.8E-04	-9.2E-04
8.0	25	1.9773	1.9756	1.9797	1.9768	-1.8E-03	2.3E-03	-4.7E-04
9.0	25	1.2800	1.2787	1.2807	1.2800	-1.3E-03	7.3E-04	2.4E-05
10.0	25	0.7697	0.7681	0.7655	0.7699	-1.6E-03	-4.2E-03	2.1E-04
11.0	25	0.4360	0.4348	0.4344	0.4362	-1.2E-03	-1.7E-03	1.6E-04
12.0	25	0.2373	0.2375	0.2379	0.2371	2.0E-04	6.7E-04	-1.1E-04
RMSE						1.6E-03	1.8E-03	9.4E-04
CPU		0.0610				2.47	2.70	0.57

This table reports the option prices, the approximation errors, the RMSE and the CPU time in the pricing of American (Panel A) and European (Panel B) put options under the Heston model. The option prices are computed by the FDM, the MCS and the RBF interpolation method. The option parameters are: $E = 10$, $\mathsf{t} = 0.25$ years and $\mathtt{r} = 10\%$. The Heston model assumes that $\kappa = 5$, $\theta = 0.16$, $\sigma = 0.9$ and $\varrho = 0.1$. The market price of risk $\zeta = 0$. The option prices are computed for stock prices $S = [8, 9, 10, 11, 12]$ and initial values of the variance $V_0 = 6.25\%$ and $V_0 = 25\%$. The benchmark for American options is given by Ikonen and Toivanen (2008), using the componentwise splitting method with a $5120 \times 2048 \times 2050$ grid. In the case of European options, the benchmark is computed with the semi-analytical formula given by Heston (1993, equations (17) and (18) on page 331) plus the put-call parity condition. The RBF interpolation is performed with 200 centers and 100 time steps while the MCS uses $10,000$ paths. For European options, the approximation by the operator splitting FDM is carried out with a $160 \times 64 \times 64$ grid. The option prices, the approximation errors and the RMSE are defined in terms of a monetary unit (e.g. $ US dollar). The CPU time is reported in seconds.

Table 9. Heston Model: Comparison of Option Prices of Recent Studies

(A) American Options Prices with V=0.0625

Reference	Stock Prices					
	8	9	10	11	12	RMSE
RBF Interpolation	1.99999	1.10701	0.51992	0.21367	0.08190	
Zvan et al. (1998)	2.00000	1.10760	0.52020	0.21380	0.08210	3.1E-04
Oosterlee (2003)	2.00000	1.10700	0.51700	0.21200	0.08150	1.5E-03
Ikonen and Toivanen (2004)	2.00000	1.10751	0.51904	0.21294	0.08181	5.6E-04
Ikonen and Toivanen (2008)	2.00000	1.10764	0.52003	0.21367	0.08204	2.9E-04
Persson and Sydow (2010)	1.99976	1.10768	0.51873	0.21424	0.08193	6.7E-04

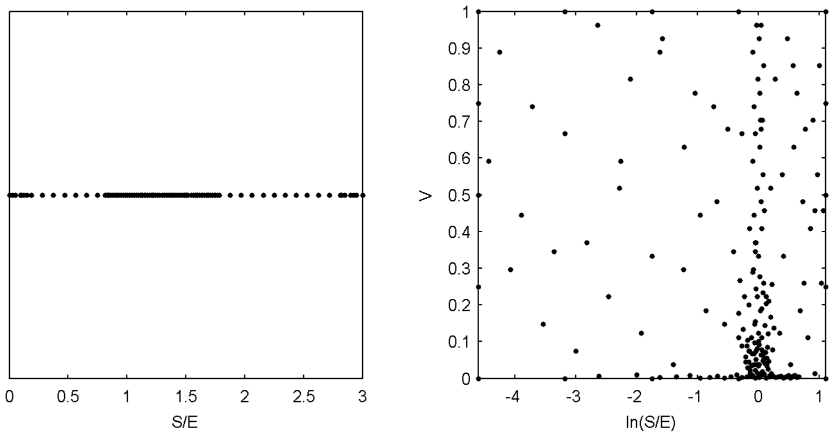
(B) American Options Prices with V=0.25

Reference	Stock Prices					
	8	9	10	11	12	RMSE
RBF Interpolation	2.07803	1.33321	0.79570	0.44812	0.24274	
Zvan et al. (1998)	2.07840	1.33370	0.79610	0.44830	0.24280	3.4E-04
Oosterlee (2003)	2.07900	1.33400	0.79600	0.44900	0.24300	7.1E-04
Ikonen and Toivanen (2004)	2.07846	1.33360	0.79585	0.44813	0.24271	2.7E-04
Ikonen and Toivanen (2008)	2.07838	1.33365	0.79598	0.44828	0.24282	2.9E-04
Persson and Sydow (2010)	2.07770	1.33219	0.79378	0.44621	0.24170	1.4E-03

This table presents the prices of American put options reported by recent studies in the literature. These prices are compared with the results of the RBF interpolation. The option prices are reported for $S = [8, 9, 10, 11, 12]$ and the initial values of the variance $V_0 = 6.25\%$ (Panel A) and $V_0 = 25\%$ (Panel B). The RMSE is computed taken as benchmark the solution by the RBF interpolation with 1000 centers and 100 time steps. The option prices and the RMSE are defined in terms of a monetary unit (e.g. $ US dollar).

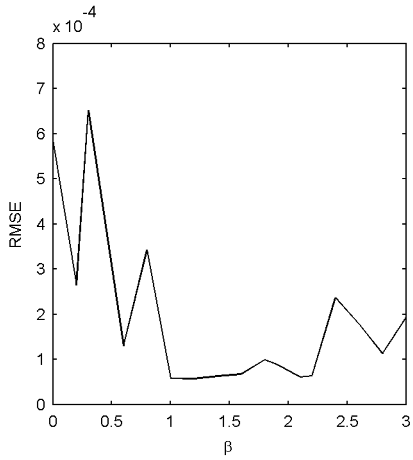
Figure 1. Centers used by the RBF Interpolation Method

(A) CEV Model **(B)** Heston Model



This figure shows the location of the centers used by the RBF interpolation method in the pricing of options under the CEV model (Panel A) and the Heston model (Panel B). Each plot contains 200 centers. The Halton sequence is used to allocate the points on the areas of interest. The centers in the (one-dimensional) CEV model are displayed in the domain $[0, 3]$. Most of the points are concentrated close to the moneyness $\frac{S}{E} = 1$. In the Heston model, we consider the domain $\left[\ln \frac{Smin}{E}, \ln \frac{Smax}{E}\right] \times [V_{Min}, V_{Max}]$ with $[\ln(1\text{E-}2), \ln(3)] \times [0, 1]$. Most of the points are allocated close to the logarithm of the moneyness $ln\left(\frac{S}{E}\right) = 0$ and the boundary $V = 0$.

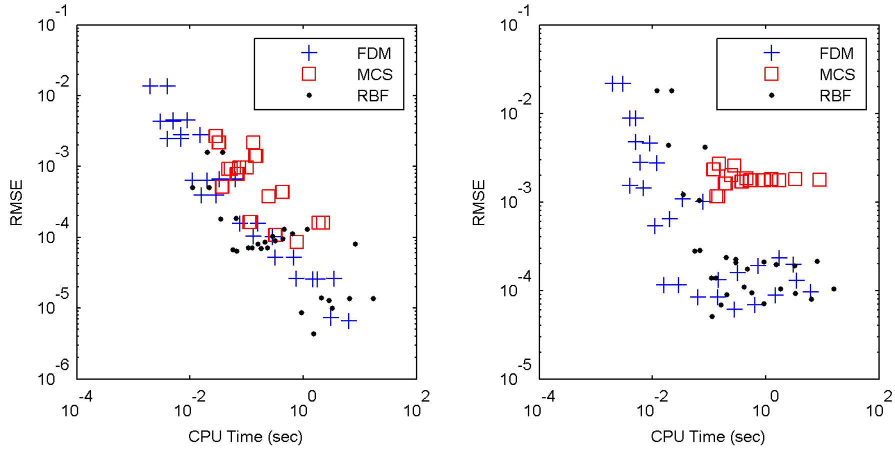
Figure 2. Accuracy of the CEV Option Prices for Different Values of β



This figure plots the RMSE of the approximation of American options by the RBF interpolation. The experiment is carried out for values of the elasticity of variance β in the interval $[0, 3]$. The benchmark is the approximation by the FDM with a 500×500 grid. The RMSE is computed with option prices that correspond to stock values $S = [8, 9, 10, 11, 12]$. The RBF interpolation method is performed with 200 centers and 200 time steps. The RMSE is defined in terms of a monetary unit (e.g. \$ US dollar).

Figure 3. CEV Option Prices: Accuracy vs. Efficiency

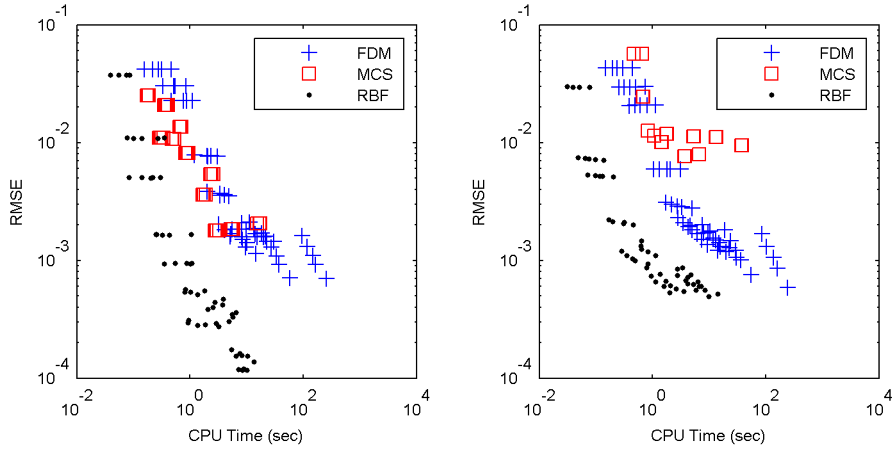
(A) European Option **(B)** American Option



This figure compares the RMSE and the CPU time of the approximation of European (Panel A) and American (Panel B) put option prices by the FDM, the MCS and the RBF interpolation method. The option parameters are: $E = 10$, $\mathfrak{t} = 0.25$ years, $\mathfrak{r} = 10\%$, $\tilde{\sigma} = 25\%$ and $\beta = 1$. The experiment considers several configurations of the grid size for the FDM, the number of paths for the MCS, and the number of centers and time steps for the RBF interpolation. The benchmark for American options is the approximation by the FDM with a 500×500 grid. For European options, the benchmark is the analytical solution given in Wong and Zhao (2008, pg. 2186, equations (2.2), (2.3) and (2.4)). The RMSE is presented in terms of a monetary unit (e.g. $ US dollar). The CPU time is reported in seconds.

Figure 4. Heston Option Prices: Accuracy vs. Efficiency

(A) European Option **(B)** American Option



This figure shows the pairs RMSE and CPU time employed in the pricing of European (Panel A) and American (Panel B) put options by the FDM, the MCS and the RBF interpolation. The option parameters are: $E = 10$, $\mathfrak{t} = 0.25$ years and $\mathfrak{r} = 10\%$. The Heston model assumes that $\kappa = 5$, $\theta = 0.16$, $\sigma = 0.9$. and $\varrho = 0.1$. The market price of risk $\zeta = 0$. The option prices are computed for stock prices $S = [8, 9, 10, 11, 12]$ and initial values of the variance $V_0 = 6.25\%$ and $V_0 = 25\%$. The experiment considers several configurations of the grid size for the FDM, the number of paths for the MCS, and the number of centers and time steps for the RBF interpolation method. The benchmark for American options is given by Ikonen and Toivanen (2008) using the componentwise splitting method with a $5120 \times 2048 \times 2050$ grid. In the case of European options, the benchmark is computed with the semi-analytical formula given by Heston (1993, p. 331, equations (17) and (18)) plus the put-call parity condition. The RMSE is presented in terms of a monetary unit (e.g. $ US dollar). The CPU time is reported in seconds.