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General Multivariate Dependence Using Associated Copulas

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GENERAL MULTIVARIATE DEPENDENCE USING ASSOCIATED COPULAS

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Abstract:

- This paper studies the general multivariate dependence and tail dependence of a random vector. We analyse the dependence of variables going up or down, covering the 2^d orthants of dimension d and accounting for non-positive dependence. We extend definitions and results from positive to general dependence using the associated copulas. We study several properties of these copulas and present general versions of the tail dependence functions and tail dependence coefficients. We analyse the perfect dependence models and elliptical copulas. We introduce the monotonic copulas and prove that the multivariate Student's t copula accounts for all types of tail dependence simultaneously.

Key-Words:

- *Non-positive Dependence; Tail Dependence; Copula Theory; Perfect Dependence Models; Elliptical Copulas*

AMS Subject Classification:

- 62H20, 60G70.

1. INTRODUCTION

A great deal of literature has been written on the analysis of the dependence structure between random variables. There is an increasing interest in the understanding of the dependencies between extreme values in what is known as tail dependence. However, the analysis of multivariate tail dependence in copula models has been exclusively focused on the positive case. Only the lower and upper tail dependence have been considered, leaving a void in the analysis of dependence structure implied by the use of these models. In this paper we tackle this issue by considering the dependence in the 2^d different orthants of dimension d for a random vector.

We define the necessary concepts to measure a general type of tail dependence in the multivariate case. We use a copula approach and base our study on the associated copulas (see [14], p. 15).

The dependence structure of time series has been studied for a long time, traditionally through the use of the Pearson's correlation coefficient. More recently, copula based measures such as the Spearman's ρ and Kendall's τ have also been used to assess concordance. Due to drawbacks of these measures when it comes to tail dependence, new methodologies have been developed. In particular the use of the tail dependence coefficient (TDC) and the tail dependence function has proven to be the way forward (see [22], [14] and [20], Chapter 5).

In the multivariate case positive tail dependence in copula models and random vectors has been analysed via the L (lower) and U (upper) TDCs and tail dependence functions, see e.g. [14], [22] or [23]. The concepts introduced in this paper are used to analyse the whole dependence structure (and not only positive) implied by the perfect dependence models and elliptical copulas.

In order to address non-positive dependence, we introduce the concept of general dependence **D** and its corresponding **D**-probability function. It is through these functions that copula theory can be extended to account for non-positive dependence. We prove that the copulas that link **D**-probability functions and its marginals are the associated copulas. All the results presented regarding general dependence are also a contribution of this work. This includes the relationship among associated copulas, the monotonic copulas, the associated elliptical copulas and the associated tail dependence function of the Student's t copula model.

The reminder of this work is divided in four sections: In the second section we present a motivation for the use of the associated copulas for non-positive dependence analysis. In the third section we present the concepts we use to study dependence in all the orthants. This includes general dependence, probability functions, associated copulas, tail dependence functions and TDCs. We obtain an equation for the relationship among all associated copulas and present three propositions regarding these copulas. In the fourth section we study the whole dependence and tail dependence structure implied by perfect dependence models and the elliptical copulas. We then obtain an expression for the associated tail dependence functions of the Student's t copula model. This model accounts for all 2^d types of tail dependence simultaneously. Finally, in the fifth section, we conclude and discuss future lines of research for general dependence.

2. MOTIVATION

The study of the associated copulas to analyse non-positive tail dependence comes as a generalisation of the use of the copula and the survival copula for lower and upper tail dependence respectively. We know briefly discuss the history of the use of the survival copula to analyse upper tail dependence. In the context of nonparametric statistics, it is possible to measure upper tail dependence by using negative transformations on all of the time series and a measure for lower tail dependence. However, presenting a formal definition of upper tail dependence in the multivariate case and analysing it in copula models can not be achieved by the use of transformations. In the bivariate case it is possible to define the upper tail dependence coefficient in terms of the copula, see e.g. [22]. In higher dimensions this definition becomes more and more cumbersome, this is simplified by the use of the survival copula, see e.g. [16]. By using the survival copula, the results and analysis of lower tail dependence via the copula have been generalised to upper tail dependence. Hence, the use of the survival copula makes the definition and study of upper tail dependence analogous to lower tail dependence. In particular, this simplifies the analysis of the upper tail dependence implied by symmetric copula models, see [23]. Similarly, in the case of nonparametric estimation of upper tail dependence, the empirical survival copula is used, see e.g. [26]. This approach avoids the use of negative transformations on the data and makes use of the theory on empirical copulas. Because of all of this, the definition and study of upper tail dependence has been derived via the use of the survival copula, see [15], [16], [20] and [26]. A whole theory on survival copulas can be found in statistical literature, see [10].

The copula and survival copula are used to analyse positive tail dependence. The study of non-positive tail dependence is also relevant when dealing with empirical data and in copula models analysis. Negative tail dependence can be found in statistical literature, particularly in the study of financial time series, see e.g. [30] and [4]. In the case of copula models, the study of tail dependence helps in the understanding of the underlying assumptions implied by the use of these models. This study has often been restricted to the positive case with the analysis of the lower and upper tail dependence coefficients. The non-positive dependence structure implied by the use of these models is hence overlooked. Failing to account for the whole dependence structure in these models can lead to undesirable consequences. For example, the Student's t copula is often used to model data with only positive tail dependence. However, although this model accounts for the positive tail dependence, it also assumes the existence of negative tail dependence. Similarly, portfolios often have stocks with positive tail dependence to maximise revenue. Simultaneously negative tail dependence is also present between these stocks in order to absorb shocks and hedge extreme falls of prices. The use of negative transformations before fitting a model with only positive tail dependence would not account for the whole tail dependence structure in the portfolio. Before choosing a copula model in these two examples it is fundamental to know the dependence structure of the data and the one implied by the model.

Associated copulas have already been used to analyse non-positive transformations (although the name is not mentioned therein), see [22] and [5]. [28] considered these copulas and concordance measures to analyse multivariate non-positive dependence. In this work we use associated copulas to define and analyse non-positive types of dependence and tail dependence. The reasoning behind this is the same as for the use of the survival copula for upper tail dependence analysis. Similarly to that case, the definition and study of non-positive tail dependence is simplified by the use of associated copulas. With the concepts introduced in this work it is possible to analyse non-positive tail dependence in parametric and nonparametric contexts. We obtain several results for general dependence and associated copulas. In the case of copula models we analyse the perfect dependence models and elliptical copulas. This can be extended to other copula models such as the Archimedean and vine copulas and to the analysis of nonparametric models for empirical data.

3. ASSOCIATED COPULAS, TAIL DEPENDENCE FUNCTIONS AND TAIL DEPENDENCE COEFFICIENTS

In this section we analyse the dependence structure among random variables using copulas. Given a random vector $\mathbf{X} = (X_1, ..., X_d)$, we use the corresponding copula C and its associated copulas to analyse its dependence structure. For this we introduce a general type of dependence $\mathbf{D}$, one for each of the 2^d different orthants. This corresponds to the lower and upper movements of the different variables.

To analyse different dependencies, we introduce the $\mathbf{D}$-probability function and present a version of Sklar's theorem that states that an associated copula is the copula that links this function and its marginals. We present a formula to link all associated copulas and three results on monotone functions and associated copulas. We then introduce the associated tail dependence function and the associated tail dependence coefficient (TDC) for the type of dependence $\mathbf{D}$. These functions generalise the positive (lower and upper) cases (extensively studied in [13, 14, 23]). With the concepts studied in this section, we aim to provide the tools to analyse the whole dependence structure among random variables, including non-positive dependence.

3.1. Copulas and dependence

The concept of copula was first introduced by Sklar [27], and is now a cornerstone topic in multivariate dependence analysis (see [14, 22, 20]). We now present the concepts of copula, general dependence and associated copulas that are fundamental for the rest of this work.

Definition 3.1. A multivariate copula $C(u_1, ..., u_d)$ is a distribution function on the d-dimensional-square $[0,1]^d$ with standard uniform marginal distributions.

If C is the distribution function of $\mathbf{U} = (U_1, ..., U_d)$, we denote as $\widehat{C}$ the distribution function of $(1 - U_1, ..., 1 - U_d)$. In the multivariate case, C is used to link multivariate distribution functions with their corresponding marginal distributions, accordingly we refer to C as the distributional copula. On the other hand, $\widehat{C}$ is used to link multivariate survival functions with their corresponding marginal survival functions, this copula is known as the survival copula.[1] The survival copula $\widehat{C}$ must not be confused with the survival function $\overline{C}(u_1, ..., u_d) = \widehat{C}(1 - u_1, ..., 1 - u_d)$. Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with joint distribution function F and marginals F_i for $i \in \{1, ...d\}$. Sklar's theorem guarantees the existence and uniqueness of a copula C

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)), \tag{3.1}$$

(see [14]). Similarly, if $\mathbf{X}$ has joint survival function $\overline{F}$, and marginal survival functions $\overline{F_i}$ for $i \in \{1, ...d\}$. Sklar's theorem for survival functions implies the existence and uniqueness of $\widehat{C}$

$$\overline{F}(x_1, ..., x_d) = \widehat{C}(\overline{F}_1(x_1), ..., \overline{F}_d(x_d)), \tag{3.2}$$

(see [22]). In the next section we generalise these equations using the concept of general dependence, which we now define.

Definition 3.2. In d dimensions, we call the vector $\mathbf{D} = (D_1, ..., D_d)$ a type of dependence if each D_i is a boolean variable, whose value is either L (lower) or U (upper) for $i \in \{1, ...d\}$. We denote by Δ the set of all 2^d types of dependence.

Each type of dependence corresponds to the variables going up or down simultaneously. Tail dependence, which we define later, refers to the case when the variables go extremely up or down simultaneously. Two well known types of dependence are lower and upper dependence. Lower dependence refers to the case when all variables go down at the same time ($D_i = L$ for $i \in \{1, ..., d\}$). Upper

[1] We use the term distributional for C, to distinguish it from the other associated copulas. The notation for the survival copula corresponds to the one used in the seminal work of Joe.[14]

dependence refers to the case when all variables go up at the same time ($D_i = U$ for $i \in \{1, ..., d\}$). These two cases are examples of positive dependence and they have been extensively studied for tail dependence analysis, see e.g. [14, 22]. In the bivariate case the dependencies $\mathbf{D} = (L, U)$ and $\mathbf{D} = (U, L)$ correspond to one variable going up while the other one goes down. These are examples of negative dependence. Negative tail dependence is often present in financial time series, see [30, 4, 15]. Hence, in dimension 2 there are four types of dependence that correspond to the four quadrants. Note that, in dimension d, for each of the 2^d orthants we define a dependence $\mathbf{D}$.

Using the concept of dependence, we now present the associated copulas, see [14, Chapter 1, p. 15].

Definition 3.3. Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with corresponding copula C, which is the distribution function of the vector $(U_1, ..., U_d)$ with uniform marginals. Let Δ denote the set of all types of dependences of Definition 3.2. For $\mathbf{D} = (D_1, ..., D_d) \in \Delta$, let $\mathbf{V_D} = (V_{D_1,1}, ..., V_{D_d,d})$ with

$$V_{D_i,i} = \begin{cases} U_i & \text{if } D_i = L \\ 1 - U_i & \text{if } D_i = U \end{cases} .$$

Note that $\mathbf{V_D}$ also has uniform marginals. We call the distribution function of $\mathbf{V_D}$, which is also a copula, the associated $\mathbf{D}$-copula and denote it $C_\mathbf{D}$. We denote $\mathcal{A}_\mathbf{X} = \{C_\mathbf{D} | \mathbf{D} \in \Delta\}$, the set of 2^d associated copulas of the random vector $\mathbf{X}$.

Note that the distributional and the survival copula are $C = C_{(L,...,L)}$ and $\widehat{C} = C_{(U,...,U)}$ respectively.

3.1.1. The D-Probability function and its associated D-Copula

The distributional copula C and the survival copula $\widehat{C}$ are used to explain the lower and upper dependence structure of a random vector respectively. We use the associated $\mathbf{D}$-copula to explain the $\mathbf{D}$-dependence structure of a random vector. For this, we first present the $\mathbf{D}$-probability functions, which generalise the joint distribution and survival functions.

Definition 3.4. Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with marginal distributions F_i for $i \in \{1, ...d\}$ and $\mathbf{D} = (D_1, ..., D_d)$ a type of dependence according to Definition 3.2. Define the event $\mathcal{B}_i(x_i)$ in the following way

$$\mathcal{B}_i(x_i) = \begin{cases} \{X_i \leq x_i\} \text{ if } D_i = L \\ \{X_i > x_i\} \text{ if } D_i = U \end{cases} .$$

Then the corresponding $\mathbf{D}$-probability function is

$$F_\mathbf{D}(x_1, ..., x_d) = P\left(\bigcap_{i=1}^{d} \mathcal{B}_i(x_i)\right) .$$

We refer to

$$F_{D_i,i} = \begin{cases} F_i \text{ if } D_i = L \\ \overline{F_i} \text{ if } D_i = U \end{cases},$$

for $i \in \{1, ...d\}$ as the marginal functions of $F_{\mathbf{D}}$ (Note that the marginals are either univariate distribution or survival functions).

In the bivariate case for example, there are four $\mathbf{D}$-probability functions: $F(x_1, x_2)$, $\overline{F}(x_1, x_2)$, $F_{LU}(x_1, x_2) = P(X_1 \leq x_1, X_2 > x_2)$ and $F_{UL}(x_1, x_2) = P(X_1 > x_1, X_2 \leq x_2)$. In general, these functions complement the use of the joint distribution and survival functions in our analysis of dependence in the 2^d orthants.

The copula we consider to analyse the $\mathbf{D}$-dependence is $C_{\mathbf{D}}$ that links the functions in Definition 3.4 with their corresponding marginals. Given that by applying decreasing transformations to a part of the data we can account for negative dependence the copulas of the $\mathbf{D}$-probability functions correspond to the associated copulas of Definition 3.3. The following theorem presents the associated copula $C_{\mathbf{D}}$ in terms of the $F_{\mathbf{D}}$ and its marginals. We restrict the proof to the continuous case (for Sklar's theorem for distribution functions see [20, 14, 22].

Theorem 3.1. ***Sklar's theorem for D-probability functions and associated copulas***

Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector, $\mathbf{D} = (D_1, ..., D_d)$ a type of dependence, $F_{\mathbf{D}}$ its $\mathbf{D}$-probability function and $F_{D_i,i}$ for $i \in \{1, ...d\}$ the marginal functions of $F_{\mathbf{D}}$ as in Definition 3.4. Let the marginal functions of $F_{\mathbf{D}}$ be continuous, then the associated copula $C_{\mathbf{D}} : [0, 1]^d \to [0, 1]$, satisfies, for all $x_1, ..., x_2$ in $[-\infty, \infty]$,

$$F_{\mathbf{D}}(x_1, ..., x_d) = C_{\mathbf{D}}(F_{D_1,1}(x_1), ..., F_{D_d,d}(x_d)), \tag{3.3}$$

which is equivalent to

$$C_{\mathbf{D}}(u_1, ..., u_d) = F_{\mathbf{D}}(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d)). \tag{3.4}$$

Conversely, let $\mathbf{D} = (D_1, ..., D_d)$ be a type of dependence and $F_{D_i,i}$ a univariate distribution function, if $D_i = L$, or a survival function, if $D_i = U$, $i \in \{1, ...d\}$,

(a) if $C_{\mathbf{D}}$ is a copula, then $F_{\mathbf{D}}$ in (3.3) defines a $\mathbf{D}$-probability function with marginals $F_{D_i,i}$, $i \in \{1, ...d\}$.

(b) if $F_{\mathbf{D}}$ is any $\mathbf{D}$-probability function, then $C_{\mathbf{D}}$ in (3.4) is a copula.

Proof: The proof of this theorem is analogous to the proof of Sklar's theorem in the continuous case. In this case for any distribution function F_i, we have that the events $\{X_i \le x_i\} \overset{P}{\sim} \{F_i(X_i) \le F_i(x_i)\}$ and $\{X_i > x_i\} \overset{P}{\sim} \{\overline{F}_i(X_i) \le \overline{F}_i(x_i)\}$. This implies

$$P(\mathcal{B}_i(x_i)) = P(F_{D_i,i}(X_i) \le F_{D_i,i}(x_i)), \tag{3.5}$$

for $i \in \{1, ..., d\}$.

Considering equation (3.5) and Definition 3.4, we have that for any $x_1, ..., x_d$ in $[-\infty, \infty]$

$$F_{\mathbf{D}}(x_1, ..., x_d) = P(F_{D_1,1}(X_1) \le F_{D_1,1}(x_1), ..., F_{D_d,d}(X_d) \le F_{D_d,d}(x_d)). \tag{3.6}$$

Using the continuity of F_i we have that $F_i(X_i)$ is uniformly distributed (see [20, Proposition (5.2 (2))]). Hence, if we define $\mathbf{U} = (F_1(X_1), ..., F_d(X_d))$, its distribution function is a copula C. Note that in this case $\mathbf{V_D}$, defined as in Definition 3.3, is equal to $(F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d))$. It follows that the distribution function of $(F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d))$ is the associated copula $C_{\mathbf{D}}$, in which case equation (3.5) implies

$$C_{\mathbf{D}}(F_{D_1,1}(x_1), ..., F_{D_d,d}(x_d)) = P(F_{D_1,1}(X_1) \le F_{D_1,1}(x_1), ..., F_{D_d,d}(X_d) \le F_{D_d,d}(x_d)),$$

and equation (3.3) follows.

If we evaluate $F_{\mathbf{D}}$ in $(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d))$, we get

$$C_{\mathbf{D}}(u_1, ..., u_d) = F_D(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d)).$$

This follows from the fact that one of the properties of the generalised inverse is that, when T is continuous, $T \circ T^{\leftarrow}(x) = x$ (see [20, Proposition (A.3)]). This equation explicitly represents $C_{\mathbf{D}}$ in terms of $F_{\mathbf{D}}$ and its marginals implying its uniqueness.

For the converse statement of the theorem, we have

(a) Let $\mathbf{U} = (U_1, ..., U_d)$ be the random vector with distribution function C. We now define

$$\begin{aligned}\mathbf{X} &= (X_1, ..., X_d)\\ &= ((F^{\leftarrow}_{D_1,1}(U_1), ..., F^{\leftarrow}_{D_d,d}(U_d))),\end{aligned}$$

and

$$\mathcal{B}_i(x_i) = \begin{cases} \{X_i \le x_i\} \text{ if } D_i = L \\ \{X_i > x_i\} \text{ if } D_i = U \end{cases},$$

for $i \in \{1, ...d\}$. Considering that $F(x) \le y \iff x \le F^{\leftarrow}(y)$, we have $\overline{F}^{\leftarrow}(x) \le y \iff x \ge \overline{F}(y)$. Using these properties, we have

$$\{U_i \le F_{D_i,i}(x_i)\} \overset{P}{\sim} \mathcal{B}_i(x_i),$$

for $i \in \{1, ...d\}$. Using this, the **D**-probability function of **X** is

$$P\left(\bigcap_{i=1}^{d} \mathcal{B}_i(x_i)\right) = C(F_{D_1,1}(x_1), ..., F_{D_d,d}(x_d)).$$

This implies that $F_{\mathbf{D}}$ defined by (3.3) is the **D**-probability function of X with marginals

$$P(\mathcal{B}_i(x_i)) = P(U_i \leq F_{D_i,i}(x_i)) = F_{D_i,i}(x_i),$$

for $i \in \{1, ...d\}$.

(b) Similarly, let $(X_1, ..., X_d)$ be the random vector with **D**-probability function $F_{\mathbf{D}}$. Define

$$\begin{aligned}\mathbf{U} &= (U_1, ..., U_d) \\ &= (F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d))\end{aligned}$$

(note that the vector is uniformly distributed). Again, using the properties of the generalised inverse, we have

$$\{U_i \leq u_i\} \overset{P}{\sim} \mathcal{B}_i(F^{\leftarrow}_{D_i,i}(u_i)),$$

for $i \in \{1, ...d\}$. Hence the distribution function of **U** is $F_{\mathbf{D}}(F^{\leftarrow}_{D_1,1}(u_1), ..., F^{\leftarrow}_{D_d,d}(u_d))$, which implies that the function is a copula.

For the properties of the generalised inverse function used in this proof, see [20, Proposition (A.3)].

□

For this theorem we referred to generalised inverses rather than inverse functions, as the first are more general. However throughout this work, whenever we are not proving a general property, we assume the distribution functions have inverse functions.

Note that this theorem implies that in the continuous case $C_{\mathbf{D}}$ is the **D**-probability function of $(F_{D_1,1}(X_1), ..., F_{D_d,d}(X_d))$ characterised in (3.3). This theorem implies the importance of the associated copulas to analyse dependencies. It also implies the Fréchet bounds for the **D**-probability functions of Definition 3.4. The bounds can also be obtained similarly to [14, Theorems 3.1 and 3.5],

$$\begin{aligned}\max\{0, F_{D_1,1}(x_1) + ... + F_{D_d,d}(x_d) - (d-1)\} \leq\ & F_{\mathbf{D}}(x_1, ..., x_d) \leq \\ & \min\{F_{D_1,1}(x_1), ..., F_{D_d,d}(x_d)\}. \end{aligned} \tag{3.7}$$

3.1.2. Properties of the associated copulas

In the bivariate case, Joe [14, Chapter 1], and Nelsen[22, Chapter 2], presented the expressions to link the associated copulas with the distributional copula C. In the multivariate case Joe [15, Equation 8.1] and Georges et al. [10, Theorem 3], presented the expression between the distributional and the survival copula and Embrechts et al. [5, Theorem 2.7] proved that is possible to express the associated copulas in terms of the distributional copula C. We now present a general equation for the relationship between any two associated copulas $C_{\mathbf{D}^*}$ and $C_{\mathbf{D}^+}$ in the multivariate case. The equation is based on all the subsets of the indices where the $\mathbf{D}^*$ and $\mathbf{D}^+$ are different. We then prove a property of associated copulas regarding exchangeability. After that, we show that the associated copulas are invariant under strictly increasing transformations and characterise the distributional copula of a random vector after strictly monotone transformations.

Proposition 3.1. *Let* $\mathbf{X} = (X_1, ..., X_d)$ *be a random vector with associated copulas* $\mathcal{A}_{\mathbf{X}}$ *and* $\mathbf{D}^* = (D_1^*, ..., D_d^*)$ *and* $\mathbf{D}^+ = (D_1^+, ..., D_d^+)$ *any two types of dependence. Consider the following sets and notations:* $I = \{1, ..., d\}$; $I_1 = \{i \in I | D_i^* = D_i^+\}$ *and* $I_2 = \{i \in I | D_i^* \neq D_i^+\}$; $d_1 = |I_1|$ *and* $d_2 = |I_2|$; $S_j = \{\text{the subsets of size } j \text{ of } I_2\}$ *and* $S_{j,k} = \{\text{The } k\text{-th element of } S_j\}$ *for* $j \in \{1, ..., d_2\}$ *and* $k \in \left\{1, ..., \binom{d_2}{j}\right\}$*. We define* $S_0 = \emptyset$ *and* $S_{0,1} = \emptyset$*; for each* $S_{j,k}$ *define* $\mathbf{W}_{j,k} = (W_{j,k.1}, ..., W_{j,k,d})$ *with*

$$W_{j,k,i} = \begin{cases} u_i & \text{if } i \in I_1 \\ 1 - u_i & \text{if } i \in S_{j,k} \\ 1 & \text{if } i \notin I_1 \cup S_{j,k} \end{cases},$$

for $i \in \{1, ...d\}$, $j \in \{0, ..., d_2\}$ *and* $k \in \left\{1, ..., \binom{d_2}{j}\right\}$.

Then the associated $\mathbf{D}^*$*-copula* $C_{\mathbf{D}^*}$ *is expressed in terms of the* $\mathbf{D}^+$*-copula* $C_{\mathbf{D}^+}$ *according to the following equation*

$$C_{\mathbf{D}^*}(u_1, ..., u_d) = \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^+}(\mathbf{W}_{j,k}). \tag{3.8}$$

Note that in the cases when at least a 1 appears in $\mathbf{W}_{j,k}$, $C_{\mathbf{D}^+}(\mathbf{W}_{j,k})$ *becomes a marginal copula of* $C_{\mathbf{D}^+}$.

Proof: Throughout this proof, it must be borne in mind that $C_{\mathbf{D}^*}$ is the distribution function of the random vector $\mathbf{V}_{\mathbf{D}^*}$ and $C_{\mathbf{D}^+}$ of $\mathbf{V}_{\mathbf{D}^+}$, defined according to Definition 3.3. Note that, for $i \in I_2$, $V_{D_i^*,i} = 1 - V_{D_i^+ i}$ and they are equal otherwise.

In the case $d_2 = 0$, we have $\mathbf{D}^* = \mathbf{D}^+$, $j \in \{0\}$ and $k \in \{1\}$[2], hence (3.8) holds. We prove (3.8) by induction on d, the dimension; it can also be proven by induction on d_2, the number of elements in which $D_i^* \neq D_i^+$. Note that in dimension $d = 1$, a copula becomes the identity function. If $D_1^* \neq D_1^+$, the expression becomes $u_1 = 1 - (1 - u_1)$; the case $D_1^* = D_1^+$ has already been covered in $d_2 = 0$, and expression (3.8) holds.

Now, suppose we are in dimension d, we prove the formula works provided it works in dimension $d-1$. We obtain an expression for $C_{\mathbf{D}^*}(u_1, ..., u_d)$ using the induction hypothesis. Consider the dependencies, on the $(d-1)$-dimension, $\mathbf{F}^* = (D_1^*, ..., D_{d-1}^*)$ and $\mathbf{F}^+ = (D_1^+, ..., D_{d-1}^+)$. We use an apostrophe on the sets and notations of $\mathbf{F}^*$ and $\mathbf{F}^+$ to differentiate them from those of $\mathbf{D}^*$ and $\mathbf{D}^+$. It follows that $d' = d - 1$ and $I' = I - \{d\}$. By the induction hypothesis, equation (3.8) holds to express $C_{\mathbf{F}^*}$ in terms of $C_{\mathbf{F}^+}$. In terms of probabilities this is equivalent to

$$\begin{aligned}&P(V_{D_i^*,1} \leq u_1, ..., V_{D_{d-1}^*,d-1} \leq u_{d-1}) \\ &= \sum_{j=0}^{d_2-1} (-1)^j \sum_{k=1}^{\binom{d_2-1}{j}} P(V_{D_1^+,1} \leq W'_{j,k,1}, ..., V_{D_{d-1}^+,d-1} \leq W'_{j,k,d-1}),\end{aligned} \tag{3.9}$$

Now there are two cases to consider depending on whether D_d^* is equal to D_d^+ or not.

Case 1. $D_d^* = D_d^+$.

In this case, it follows that, $I_1' = I_1 - \{d\}$, $I_2' = I_2$, $d_2' = d_2$ *and* $V_{D_d^*,d} = V_{D_d^+,d}$. *If we intersect the events in equation (3.9) with the event* $\{V_{D_d^*d} \leq u_d\}$ *we get*

(3.10)

$$\begin{aligned}&P(V_{D_i^*,1} \leq u_1, ..., V_{D_{d-1}^*,d-1} \leq u_{d-1}, V_{D_d^*d} \leq u_d) \\ &= \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} P(V_{D_1^+,1} \leq W'_{j,k,1}, ..., V_{D_{d-1}^+,d-1} \leq W'_{j,k,d-1}, V_{D_d^+,d} \leq u_d).\end{aligned}$$

Because $I_2' = I_2$, *in this case, for* $j \in \{1, ..., d_2\}$ *and* $k \in \left\{1, ..., \binom{d_2}{j}\right\}$, *the events* $S'_{j,k}$ *are equal to* $S_{j,k}$. *Considering this, and* $I_1' = I_1 - \{d\}$, *we have*

$$(\mathbf{W}'_{j,k}, u_d)_i = W_{j,k,i}$$

for $i \in \{1, ..., d\}$, *so* $(\mathbf{W}'_{j,k}, u_d) = \mathbf{W}_{j,k}$ *for* $j \in \{1, ..., d_2\}$ *and* $k \in \left\{1, ..., \binom{d_2}{j}\right\}$. *Equation (3.10) then implies:*

$$C_{D^*}(u_1, ..., u_d) = \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{D^+}(\mathbf{W}_{j,k}).$$

Case 2. $D_d^* \neq D_d^+$

In this case, it holds that, $I_1' = I_1$, $I_2' = I_2 - \{d\}$, $d_2' = d_2 - 1$. *We want to obtain*

[2]Note that we are using the convention $0! = 1$

an expression for $C_{\mathbf{D}^*}(u_1,...,u_d) = P(V_{D_i^*,1} \leq u_1,...,V_{D_d^*,d} \leq u_d)$*, using the induction hypothesis. Considering that, in general,* $P(A) = P(A\cap B)+P(A\cap B^c)$ *we have that*

$$P(V_{D_i^*,1} \leq u_1,...,V_{D_{d-1}^*,d-1} \leq u_{d-1}) = P(V_{D_i^*,1} \leq u_1,...,V_{D_{d-1}^*,d-1} \leq u_{d-1}, V_{D_d^*,d} \leq u_d)$$

$$+P(V_{D_i^*,1} \leq u_1,...,V_{D_{d-1}^*,d-1} \leq u_{d-1}, V_{D_d^*,d} \geq u_d),$$

which implies

(3.11)

$$C_{\mathbf{D}^*}(u_1,...,u_d) = P(V_{D_1^*,1} \leq u_1,...,V_{d-1}^* \leq u_{d-1}) - P(V_{D_1^*,1} \leq u_1,...,V_{d-1}^* \leq u_{d-1}, V_{D_d^*,d} \geq u_d).$$

Note that, in this case $V_{D_d^*,d} = 1-V_{D_d^+,d}$*. This implies that the event* $\{V_{D_d^*,d} \geq u_d\}$ *is equivalent to* $\{V_{D_d^+,d} \leq 1-u_d\}$*. If we intersect the events involved in equation (3.9) with the event* $\{V_{D_d^*,d} \geq u_d\}$ *we get*

(3.12)

$$P(V_{D_1^*,1} \leq u_1,...,V_{D_{d-1}^*,d-1} \leq u_{d-1}, V_{D_d^*,d} \geq u_d) =$$
$$\sum_{j=0}^{d_2-1}(-1)^j\sum_{k=1}^{\binom{d_2-1}{j}} P(V_{D_1^+,1} \leq W'_{j,k,1},...,V_{D_{d-1}^+,d-1} \leq W'_{j,k,d-1}, V_{D_d^+,d} \leq 1-u_d).$$

Combining equations (3.9), (3.11) and (3.12), we obtain

(3.13)

$$C_{\mathbf{D}^*}(u_1,...,u_d) = \sum_{j=0}^{d_2-1}(-1)^j\sum_{k=1}^{\binom{d_2-1}{j}} C_{\mathbf{D}^+}(\mathbf{W}'_{j,k},1) - \sum_{j=0}^{d_2-1}(-1)^j\sum_{k=1}^{\binom{d_2-1}{j}} C_{\mathbf{D}^+}(\mathbf{W}'_{j,k},1-u_d).$$

Note that, in this case, the sets I_2 *and* I'_2 *satisfy* $I_2 = I'_2 \cup \{d\}$*.*

The rest of the proof is based on the fact that for $j \in \{1,...,d-1\}$ the elements of size j of I_2 are the elements of size j of I'_2 plus the elements of size $j-1$ of I'_2 attaching them $\{d\}$. Considering our notation, this means

$$S_j = S'_j \cup S''_{j-1}, \tag{3.14}$$

with $S''_{j-1} = \left\{S''_{j-1,k} = S'_{j-1,k} \cup \{d\} \middle| \, k \in \left\{1,...,\binom{d_2}{j}\right\}\right\}$ for $j \in \{1,...,d-1\}$. Further to this, by definition of $\mathbf{W}_{j,k}$ we have the following three equalities

$$(\mathbf{W}'_{j,k},1)_i = \begin{cases} u_i & \text{if } i \in I_1 \\ 1-u_i & \text{if } i \in S'_{j,k} \\ 1 & \text{if } i \notin I_1 \cup S'_{j,k} \end{cases}, \quad W_{j,k,i} = \begin{cases} u_i & \text{if } i \in I_1 \\ 1-u_i & \text{if } i \in S_{j-1,k} \\ 1 & \text{if } i \notin I_1 \cup S_{j,k} \end{cases}$$

$$\text{and } (\mathbf{W}'_{j-1,k},1-u_d)_i = \begin{cases} u_i & \text{if } i \in I_1 \\ 1-u_i & \text{if } i \in S''_{j-1,k} \\ 1 & \text{if } i \notin I_1 \cup S''_{j-1,k} \end{cases},$$

for $i \in \{1,...d\}$, $j \in \{1,...,d-1\}$ and $k \in \left\{1,...,\binom{d_2}{j}\right\}$. These three equalities and equation (3.14) imply that, for a fixed j, if we sum $C_{\mathbf{D}^+}$ evaluated in all of the $(\mathbf{W}'_{j,k}, 1)$ and $(\mathbf{W}'_{j,k}, 1-u_d)$ for different k, we get the sum of $C_{\mathbf{D}^+}$ evaluated on $\mathbf{W}_{j,k}$ for different k, that is:

$$\sum_{k=1}^{\binom{d_2-1}{j}} C_{\mathbf{D}^+}(\mathbf{W}'_{j,k}, 1) + \sum_{k=1}^{\binom{d_2-1}{j-1}} C_{\mathbf{D}^+}(\mathbf{W}'_{j-1,k}, 1-u_d) = \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^+}(\mathbf{W}_{j,k}), \tag{3.15}$$

for $j \in \{1, ..., d-1\}$. Also, the equalities

$$(\mathbf{W}'_{0,1}, 1)_i = W_{0,1,i} \text{ and } (\mathbf{W}'_{d-1,1}, 1-u_d)_i = W_{d,1,i},$$

hold for $i \in \{1, ...d\}$; the result is implied by these two equalities and equations (3.13) and (3.15). □

Note that this expression is mainly dependent on the subsets of I_2, the elements in which $D_i^* \neq D_i^+$. Because of this, the expression is reflexible, meaning that it yields the same formula to express $C_{\mathbf{D}^+}$ in terms of $C_{\mathbf{D}^*}$. The formula has 2^{d_2} elements (one for every $S_{j,k}$). In particular, equation (3.8) can be used to express any associated copula in terms of the distributional copula C. This is useful considering that the expression found in literature for copula models is the one for the distributional copula.

Corollary 3.1. *Let $X = (X_1, ..., X_d)$ be a random vector with copula C and $\mathbf{D} = (D_1, ..., D_d)$ a type of dependence. Consider the same notations of proposition (3.1) with $I_1 = \{i \in I | D_i = L\}$ and $I_2 = \{i \in I | D_i = U\}$. Then the associated $\mathbf{D}$-copula $C_{\mathbf{D}}$ is expressed in terms of C according to*

$$C_{\mathbf{D}}(u_1, ..., u_d) = \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C(\mathbf{W}_{j,k}).$$

It must be borne in mind that the fact that a particular model does not account for a type of tail dependence does not mean it can not be used to model it. Using the relationship among associated copulas, as long as a model accounts for one type of tail dependence, it can be used to model an arbitrary type of dependence. For example consider a copula C that accounts only for lower tail dependence. If we want to model $\mathbf{D}$-tail dependence we can assume that C is the associated $\mathbf{D}$-copula.

In order to analyse the symmetry and exchangeability of copula models, we use the following definitions.

Definition 3.5. Let $\mathbf{D} = (D_1, ..., D_d)$ be a type of dependence, the complement dependence is defined as $\mathbf{D}^{\complement} = (D_1^{\complement}, ..., D_d^{\complement})$, with

$$D_i^{\complement} = \begin{cases} U \text{ if } D_i = L \\ L \text{ if } D_i = U \end{cases},$$

for $i \in \{1, ..., d\}$. We say that the random vector X, with associated copulas $\mathcal{A}_{\mathbf{X}}$, is complement (reflection or radial) symmetric, if there exists $\mathbf{D}^* \in \Delta$, such that $C_{\mathbf{D}^*} = C_{\mathbf{D}^{*\complement}}$.

Definition 3.6. A random vector $X = (X_1, ..., X_d)$ is said to be exchangeable if, for every permutation PR of $\{1, ..., d\}$, $PR(i) = p_i$, it holds that $(X_1, ..., X_d) \stackrel{d}{=} (X_{p_1}, ..., X_{p_d})$. A copula C is said to be exchangeable if it is the distribution function of an exchangeable vector, in which case, the copula satisfies $C(u_1, ..., u_d) = C(u_{p_1}, ..., u_{p_d})$ for every permutation. The term permutation symmetric is also used for this property.

In the following proposition we obtain equivalences for the exchangeability and equalities regarding associated copulas. Note that in Definition 3.5 we defined complement symmetry when there exists $\mathbf{D}^*$, such that $C_{\mathbf{D}^*} = C_{\mathbf{D}^{*\complement}}$. The reason for this, as we now prove, is that if it holds for one dependence it holds for them all. Also, we prove that if $C_{\mathbf{D}^\circ}$ is exchangeable, then $C_{\mathbf{D}^{\circ\complement}}$ is exchangeable among other general properties.

According to proposition (3.1), the relationship between two associated copulas $C_{\mathbf{D}^*}$ and $C_{\mathbf{D}^+}$ is determined by the elements in which $\mathbf{D}^*$ and $\mathbf{D}^+$ are different. Such elements are denoted as I_2, given that we deal with several types of dependence, we denote this set as $I_2(\mathbf{D}^*, \mathbf{D}^+)$ to indicate the dependencies to which it refers. We do the same for $I_1(\mathbf{D}^*, \mathbf{D}^+)$, the elements in which the dependencies are equal.

Proposition 3.2. *Let $\mathbf{X}$ be a vector with corresponding associated copulas $\mathcal{A}_{\mathbf{X}}$, and let $\mathbf{D}^*$, $\mathbf{D}^+$, $\mathbf{D}^\circ$ and $\mathbf{D}^\times$ be types of dependencies. Then the following equivalences hold:*

(i) *If $C_{\mathbf{D}^*} \equiv C_{\mathbf{D}^+}$ and $I_2(\mathbf{D}^*, \mathbf{D}^+) = I_2(\mathbf{D}^\times, \mathbf{D}^\circ)$ then $C_{\mathbf{D}^\times} \equiv C_{\mathbf{D}^\circ}$.*
In particular $C_{\mathbf{D}^*} \equiv C_{\mathbf{D}^{*\complement}}$, for some $\mathbf{D}^*$, implies $C_{\mathbf{D}} \equiv C_{\mathbf{D}^\complement}$ for all $\mathbf{D} \in \Delta$.

(ii) *If $C_{\mathbf{D}^\circ}$ is exchangeable, then the following hold:*

(a) *$C_{\mathbf{D}^*}$ is exchangeable over the elements of $I_1(\mathbf{D}^*, \mathbf{D}^\circ)$ and over the elements of $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$.*

In particular, if $C_{\mathbf{D}^\circ}$ is exchangeable, then $C_{\mathbf{D}^{\circ\complement}}$ is exchangeable.

(b) *If $|I_2(\mathbf{D}^*, \mathbf{D}^\circ)| = |I_2(\mathbf{D}^+, \mathbf{D}^\circ)|$, let PR be any permutation of $\{1, ..., d\}$ that assigns to each element of $I_2(\mathbf{D}^+, \mathbf{D}^\circ)$, an element of $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$. Denote $i' = PR(i)$, then*

$$C_{\mathbf{D}^*}(u_1, ..., u_d) = C_{\mathbf{D}^+}(u_{1'}, ..., u_{d'}).$$

(c) *If d is even and there exists $C_{\mathbf{D}^*}$ exchangeable, such that $|I_2(\mathbf{D}^*, \mathbf{D}^\circ)| = \frac{d}{2}$ then $C_{\mathbf{D}} \equiv C_{\mathbf{D}^\complement}$ for all $\mathbf{D} \in \Delta$.*

Proof: (i) This follows from the fact $I_2(\mathbf{D}^*, \mathbf{D}^+) = I_2(\mathbf{D}^\times, \mathbf{D}^\circ) \implies I_2(\mathbf{D}^\times, \mathbf{D}^*) = I_2(\mathbf{D}^\circ, \mathbf{D}^+)$, which is easily verified considering the different cases. From proposition (3.1), we have that the vectors $\mathbf{W}_{j,k}$ are the same in both cases, which implies

$$\begin{aligned} C_{\mathbf{D}^\times}(u_1, ..., u_d) &= \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^*}(\mathbf{W}_{j,k}) \\ &= \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^+}(\mathbf{W}_{j,k}) \\ &= C_{\mathbf{D}^\circ}(u_1, ..., u_d). \end{aligned}$$

In particular, note that $I_2(\mathbf{D}^*, \mathbf{D}^{*\complement}) = I_2(\mathbf{D}, \mathbf{D}^\complement) = \{1, ..., d\}$ for every $\mathbf{D} \in \Delta$. Then $C_{\mathbf{D}^*} \equiv C_{\mathbf{D}^{*\complement}}$ implies $C_{\mathbf{D}} \equiv C_{\mathbf{D}^\complement}$ for every $\mathbf{D} \in \Delta$.

(ii) (a) From proposition (3.1) we have

$$C_{\mathbf{D}^*}(u_1, ..., u_d) = \sum_{j=0}^{d_2} (-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k}). \tag{3.16}$$

Consider $j \in \{0, ..., d_2\}$ and $k \in \left\{1, ..., \binom{d_2}{j}\right\}$, from the way it is defined, $W_{j,k,i} = u_i$ for every $i \in I_1(\mathbf{D}^*, \mathbf{D}^\circ)$. The exchangeability of $C_{\mathbf{D}^\circ}$ implies that $C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k})$ is exchangeable over $I_1(\mathbf{D}^*, \mathbf{D}^\circ)$. Hence, equation (3.16) implies that $C_{\mathbf{D}^*}$ is exchangeable over $I_1(\mathbf{D}^*, \mathbf{D}^\circ)$. Now, let $j \in \{0, ..., d_2\}$ be fixed, note that each $\mathbf{W}_{j,k}$, $k \in \left\{1, ..., \binom{d_2}{j}\right\}$, is based on a different subset of size j of $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$. Consider the sum $\sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}_{j,k})$ as a function, given that $C_{\mathbf{D}^\circ}$ is exchangeable and that the sum considers all the subsets of size j of $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$, it follows that that function is exchangeable over $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$. Equation (3.16) then implies that $C_{\mathbf{D}^*}$ is exchangeable over $I_2(\mathbf{D}^*, \mathbf{D}^\circ)$. In particular $C_{\mathbf{D}^{\circ\complement}}$ is exchangeable over $I_2(\mathbf{D}^\circ, \mathbf{D}^{\circ\complement}) = \{1, ..., d\}$.

(ii) (b) Considering proposition (3.1), to avoid confusion, in this part of the proof, we denote with a superindex $*$ all the corresponding notations to express $C_{\mathbf{D}^*}$ in terms $C_{\mathbf{D}^\circ}$ and with a superindex $+$ all the notations to express $C_{\mathbf{D}^+}$ in terms of $C_{\mathbf{D}^\circ}$. From the hypothesis we know $d_2^+ = d_2^*$, so no superindex is used for this value.

Let PR be any permutation that satisfies the hypothesis. We denote $i' = PR(i)$ and $A' = \{PR(i) | i \in A\}$ with $A \subseteq \{1, ..., d\}$. From proposition (3.1), we have

$$C_{\mathbf{D}^+}(u_{1'},...,u_{d'}) = \sum_{j=0}^{d_2}(-1)^j \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}^{+(1)}_{j,k}) \tag{3.17}$$

with

$$W^{+(1)}_{j,k,i} = \begin{cases} u_{i'} & \text{if } i \in I^+_1 \\ 1-u_{i'} & \text{if } i \in S^+_{j,k} \\ 1 & \text{if } i \notin I^+_1 \cup S^+_{j,k} \end{cases}$$

$i \in \{1,...,d\}$, and $S^+_{j,k}$ is the k-th element of size j of I^+_2, $j \in \{0,...,d_2\}$ and $k \in \left\{1,...,\binom{d_2}{j}\right\}$. Given that $C_{\mathbf{D}^\circ}$ is exchangeable, we have that

$$C_{D^\circ}(\mathbf{W}^{+(1)}_{j,k}) = C_{D^\circ}(\mathbf{W}^{(2)}_{j,k}), \tag{3.18}$$

with

$$W^{(2)}_{j,k,i} = \begin{cases} u_i & \text{if } i \in I_1(\mathbf{D}^*,\mathbf{D}^\circ) \\ 1-u_i & \text{if } i \in S^{+\prime}_{j,k} \\ 1 & \text{if } i \notin I_1(\mathbf{D}^*,\mathbf{D}^\circ) \cup S^{+\prime}_{j,k} \end{cases}$$

For each $k \in \left\{1,...,\binom{d_2}{j}\right\}$, $S^{+\prime}_{j,k}$ is a different subset of size j of $I_2(D^*,D^\circ)$. Hence,

$$\sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}^{(2)}_{j,k}) = \sum_{k=1}^{\binom{d_2}{j}} C_{\mathbf{D}^\circ}(\mathbf{W}^*_{j,k}), \tag{3.19}$$

for $j \in \{0,...,d_2\}$. Proposition (3.1) and equations (3.17) to (3.19) imply the result.

(ii) (c) Note that $|I_2(\mathbf{D}^*,\mathbf{D}^\circ)| = \frac{d}{2} \implies |I_2(\mathbf{D}^*,\mathbf{D}^{\circ\complement})| = \frac{d}{2}$. Consider any permutation of $\{1,...,d\}$, $PR(i) = i'$, that assigns to each element of $I_2(\mathbf{D}^*,\mathbf{D}^\circ)$ an element of $I_2(\mathbf{D}^*,\mathbf{D}^{\circ\complement})$. Given that $C_{\mathbf{D}^*}$ is exchangeable we can use (ii)(b)

$$C_{\mathbf{D}^{\circ\complement}}(u_1,...,u_d) = C_{\mathbf{D}^\circ}(u_{1'},...,u_{d'}).$$

Considering that $C_{\mathbf{D}^\circ}$ is exchangeable, this implies $C_{\mathbf{D}^\circ} \equiv C_{\mathbf{D}^{\circ\complement}}$. (i) then implies $C_{\mathbf{D}} \equiv C_{\mathbf{D}^\complement}$ for all $\mathbf{D} \in \Delta$.

□

Similar to a distributional copula, in the continuous case, all the associated copulas are also invariant under strictly increasing transformations. We state this in the following proposition:

Proposition 3.3. *Let T_1,..., T_d be strictly increasing functions and $\mathbf{X} = (X_1, ..., X_d)$ a random vector with corresponding distribution function and marginals, $\mathbf{D}$ a type of dependence and $\mathbf{D}$-copula $C_{\mathbf{D}}$. Then, in the continuous case,*

$$\widetilde{\mathbf{X}} = (T_1(X_1), ..., T_d(X_d))$$

also has the same corresponding $\mathbf{D}$-copula $C_{\mathbf{D}}$.

Proof: This result follows straightforwardly from the fact that the distributional copula is invariant under strictly increasing transfromations (see [20, Proposition (5.6)]) as all associated copulas are implied by this copula using Corollary 3.1. □

In the bivariate case, Nelsen [22, Theorem 2.4.4] and Embrechts et al. [5, Theorem 2.7], characterised the copula after the use of strictly monotone functions on random variables. In the multivariate case, this can be done using the associated copulas as we show in the following proposition.

Proposition 3.4. *Let T_1,..., T_d be strictly monotone functions and $\mathbf{X} = (X_1, ..., X_d)$ a random vector with corresponding distributional copula C. Then the distributional copula of $\widetilde{\mathbf{X}} = (T_1(X_1), ..., T_d(X_d))$ is the associated $\mathbf{D}$-copula $C_{\mathbf{D}}$ of $\mathbf{X}$, with*

$$D_i = \begin{cases} L \text{ if } T_i \text{ is strictly increasing} \\ U \text{ if } T_i \text{ is strictly decreasing} \end{cases},$$

for $i \in \{1, .., d\}$, whose expression is given by Corollary (3.1).

Proof: By using the inverse functions of T_i and F_i, $i \in \{1, ..., d\}$ we have:

$$T_i(X_i) \leq (\widetilde{F}_i^{\leftarrow}(u_i)) \overset{P}{\sim} \mathcal{B}_i(F_{D_i,i}^{\leftarrow}(u_i)),$$

for $i \in \{1, ..., d\}$, with $\mathcal{B}_i$ as in Definition 3.4, which implies that the distributional copula of $\widetilde{\mathbf{X}}$ is $C_{\mathbf{D}}$. □

3.2. Associated tail dependence functions and tail dependence coefficients

Considering the results obtained so far, it is possible to introduce a general definition of tail dependence function and tail dependence coefficients considering the dependence $\mathbf{D}$. For the analysis of the conditions of the existence of the tail dependence function see [21]. The general expression of the tail dependence function is the following (for the positive case, see [23])

Definition 3.7. Let $I = \{1, ..., d\}$, $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with copula C, $\mathbf{D} = (D_1, ..., D_d)$ be a type of dependence and $C_{\mathbf{D}}$ be the copula of the random vector $\mathbf{V_D}$ of Definition 3.3. For any $\emptyset \neq S \subseteq I$, let $\mathbf{D}(S)$ denote the corresponding $|S|$-dimensional marginal dependence of $\mathbf{D}$ and $C_{\mathbf{D}(S)}$ the copula of the $|S|$-dimensional marginal $\{V_{D_i,i} | i \in S\}$. Define the associated $\mathbf{D}(S)$-tail dependence functions $b_{\mathbf{D}(S)}$ of $C_{\mathbf{D}}$, $\emptyset \neq S \subseteq I$ as

$$b_{\mathbf{D}(S)}(w_i, i \in S) = \lim_{u \downarrow 0} \frac{C_{\mathbf{D}(S)}(uw_i,\ i \in S)}{u}, \forall w = (w_1, ..., w_d) \in \mathbb{R}^d_+.$$

Given that these functions come from the associated copulas, we call the set of all $\mathbf{D}$-tail dependence functions the associated tail dependence functions. When $S = \{1, ..., d\}$ we omit such subindex.

The tail dependence functions was introduced as a generalisation to the tail dependence coefficient to determine the existence of dependence among random variables, see [23, 13]. With the definition of the general tail dependence coefficient, that we now present, it is possible to determine the existence of tail dependence for a general dependence $\mathbf{D}$.

Definition 3.8. Consider the same conditions of Definition 3.7. Define the associated $\mathbf{D}(S)$-tail dependence coefficients $\lambda_{\mathbf{D}(S)}$ of $C_{\mathbf{D}}$, $\emptyset \neq S \subseteq I$ as

$$\lambda_{\mathbf{D}(S)} = \lim_{u \downarrow 0} \frac{C_{\mathbf{D}(S)}(u, .., u)}{u}.$$

We say that $\mathbf{D}(S)$-tail dependence exists whenever $\lambda_{\mathbf{D}(S)} > 0$.

Note that

$$C_{\mathbf{D}(S)}(u, .., u) = C_{\mathbf{D}}(u_1, .., u_d) \geq C_{\mathbf{D}}(u, .., u),$$

with $u_i = \begin{cases} u \text{ if } i \in S \\ 1 \text{ if } i \notin S \end{cases}$ $i \in \{1, ..., d\}$. Because of this, $\lambda_{\mathbf{D}(S)} \geq \lambda_{\mathbf{D}}$, so $\mathbf{D}$-tail dependence implies $\mathbf{D}(S)$-tail dependence for all $\emptyset \neq S \subseteq I$.

4. MODELLING GENERAL DEPENDENCE

In this section we analyse dependence in copula models. We analyse two examples, the perfect dependence cases and the elliptically contoured copulas. With this analysis, it is possible to know the general dependence and tail dependence structure implied by the use of these models. For the perfect dependence case we obtain the associated copulas of the perfect positive dependence model. We then prove that these copulas correspond to the use of strictly monotone transformations on a random variable, so we call this copulas the monotonic copulas. For the elliptical copulas we first study the elliptically contoured distributions and prove a proposition that chacaterises their corresponding associated copulas. We then present the associated tail dependence functions of the Student's t copula model. This model accounts for all 2^d types of tail dependencies. The analysis of general dependence presented in this section complements the analysis of positive tail dependence for these models.

4.1. Perfect dependence cases

We now analyse the most basic examples of copula models. They correspond to all the variables being either independent or perfectly dependent. We first present the independent copula. We then present the associated copulas of the perfect positive dependence model and prove that they correspond to the use of strictly monotone transformations on a random variable. It follows that these are the copulas of the perfect dependence models. For the independence case, let $\mathbf{U} = (U_1, ..., U_d)$ be a random vector with $\{U_i\}_{i=1}^d$ independent uniformly distributed random variables. The distribution function of U is the copula $C(u_1, ..., u_d) = \prod_{i=1}^{d} u_i$, which is known as the independence copula. It follows that the associated copula are also equal to the independence copula. This is the copula of any random vector $X = (X_1, ..., X_d)$ with $\{X_i\}_{i=1}^d$ independent random variables. For the case of perfect positive dependence, let $\mathbf{U}$ be the d-dimensional vector $\mathbf{U} = (W, ..., W)$ with W a uniform random variable. The distribution function of $\mathbf{U}$ is the copula

$$C(u_1, ..., u_d) = \min\{u_i\}_{i=1}^d. \tag{4.1}$$

This copula is the comonotonic copula. We now analyse the general associated copula $C_{\mathbf{D}}$ for this example. Let $\mathbf{D}$ be a type of dependence and $I = \{1, ..., d\}$. Define $I_L = \{i \in I | D_i = L\}$ and $I_U = \{i \in I | D_i = L\}$. From Definition 3.3, the associated $\mathbf{D}$-copula $C_{\mathbf{D}}$ is the distribution function of the vector $\mathbf{V_D}$. Let us assume that neither I_L nor I_U are empty (the other two cases have just been analysed), then the associated $\mathbf{D}$-copula is

$$C_{\mathbf{D}}(u_1, ..., u_d) = P((W \leq \min\{u_i\}_{i \in I_L}) \cap (W \geq \max\{1 - u_i\}_{i \in I_U})).$$

It follows that, for $\min\{u_i\}_{i \in I_L} > \max\{1 - u_i\}_{i \in I_U}$, this probability is equal to zero; in the other case we have

$$C_{\mathbf{D}}(u_1, ..., u_d) = \min\{u_i\}_{i \in I_L} + \min\{u_i\}_{i \in I_U} - 1.$$

Therefore, a general expression is

$$C_{\mathbf{D}}(u_1, ..., u_d) = \max\{0, \min\{u_i\}_{i \in I_L} + \min\{u_i\}_{i \in I_U} - 1\}. \tag{4.2}$$

Note that $C_{\mathbf{D}} = C_{\mathbf{D}^{\complement}}$. Hence, the d-dimensional vector $\mathbf{U} = (W, ..., W)$ is complement symmetric, according to Definition 3.5. There are 2^{d-1} different associated copulas of this vector. In the bivariate case the associated (L, U)-copula C_{LU} is equal to the Fréchet lower bound copula, also known as the countermonotonic copula.

The copulas obtained in (4.2) are a generalisation of the countermonotonic copula of the bivariate case. The countermonotonic copula is the lower Fréchet bound for copula and corresponds to perfect negative dependence. More generally, these copulas appear when modelling perfect non-positive dependence, see [20, Example 5.22]. In the following proposition we prove that, in d dimensions, the copulas of (4.1) and (4.2) correspond to the use of strictly monotone transformations on a random variable. Because of this, we call these copulas the monotonic copulas.

Proposition 4.1. *Let Z be a random variable, and let $\{T_i\}_{i=1}^d$ be strictly monotone functions, then the distributional copula of the vector $X = (T_1(Z), ...T_d(Z))$ is one of the monotonic copulas of equations (4.1) or (4.2) with $\mathbf{D} = (D_1, ..., D_d)$,*

$$D_i = \begin{cases} L \text{ if } T_i \text{ is strictly increasing} \\ U \text{ if } T_i \text{ is strictly decreasing} \end{cases}.$$

Conversely, consider a random vector $\mathbf{X} = (X_1, ..., X_d)$ whose distributional copula is a monotonic copula of equation (4.1) or (4.2) for certain $\mathbf{D}$. Then there exist monotone functions $\{T_i\}_{i=1}^d$ and a random variable Z such that

$$(X_1, ..., X_d) \stackrel{d}{=} (T_1(Z), ..., T_d(Z)), \tag{4.3}$$

the $\{T_i\}_{i=1}^d$ satisfy that T_i is strictly increasing if $D_i = L$ and strictly decreasing if $D_i = U$ for $i \in \{1, ..., d\}$. In both cases the vector $\mathbf{X}$ is complement symmetric.

Proof: Let F be the distribution function of Z. Considering the uniform random variable $F(Z)$ it is clear that the copula of the d-dimensional vector $(Z, ..., Z)$ is the Fréchet upper bound copula $\min\{u_i\}_{i=1}^d$of equation (4.1). The result is then implied by proposition (3.4).

The converse statement is a generalisation of [5, Theorem 3.1]. We have that the distributional copula of $\mathbf{X}$ is a monotonic copula for certain $\mathbf{D}$. Note that the associated $\mathbf{D}$-copula of $\mathbf{X}$ is the Fréchet upper bound copula. Let $\{\alpha_i\}_{i=1}^d$ be any invertible monotone functions that satisfy α_i is strictly increasing if $D_i = L$ and strictly decreasing if $D_i = U$ for $i \in \{1, ..., d\}$. Proposition (3.4) implies that the copula of $\mathbf{A} = (\alpha_1(X_1), ..., \alpha_d(X_d))$ is the Fréchet upper bound copula. According to Fréchet [9] and Embrechts et al. [6], there exists a random variable Z and strictly increasing $\{\beta_i\}_{i=1}^d$ such that

$$(\alpha_1(X_1), ..., \alpha_d(X_d)) \stackrel{d}{=} (\beta_1(Z), ..., \beta_d(Z)).$$

By defining $T_i = \alpha_i^{-1} \circ \beta_i$ for $i \in \{1, ..., d\}$ we get the result.

In both cases the associated copulas of $\mathbf{X}$ are the monotonic copulas implying that the vector is complement symmetric. □

This proposition means that the copula of perfect dependence models, where all variables have perfect positive or negative dependence, is a monotonic copula. Regarding tail dependence, suppose the vector $\mathbf{X}$ has distributional copula C^* equal to a monotonic copula $C_{\mathbf{D}}$ of equations (4.1) or (4.2) for certain $\mathbf{D}$. Considering Definition 3.3 of the associated copulas, this implies that $C^*_{\mathbf{D}}$ is the comonotonic copula. Hence $C^*_{\mathbf{D}}$ and $C^*_{\mathbf{D}^\complement}$ satisfy equation (4.1). It follows that the $\mathbf{D}$ and $\mathbf{D}^\complement$ tail dependence functions of the vector $\mathbf{X}$ are

$$b^*_{\mathbf{D}}(w_1, ..., w_d) = b^*_{\mathbf{D}^\complement}(w_1, ..., w_d) = \min\{w_1, ..., w_d\}.$$

The other associated copulas satisfy equation (4.2) for some $\mathbf{D}^0$. It follows that the corresponding tail dependence functions of $\mathbf{X}$ are equal to zero

4.2. Elliptically contoured distributions

We now anlayse the dependence structure of elliptically contoured distributions. We first present the definition of this model. Then we present its corresponding associated copulas. Finally we present the associated tail dependence functions of the Student's t copula model.

Elliptically contoured distributions, or elliptical distributions, were introduced by Kelker [17] and have been analysed by several authors (see for example [8, 11]). They have the following form.

Definition 4.1. The random vector $\mathbf{X} = (X_1, ..., X_d)$ has a multivariate elliptical distribution, denoted as $\mathbf{X} \sim El_d(\mu, \Sigma, \psi)$, if for $\mathbf{x} = (x_1, ..., x_d)'$ its characteristic function has the form

$$\varphi(\mathbf{x}; \mu, \Sigma) = \exp(i\mathbf{x}'\mu)\psi_d(\frac{1}{2}\mathbf{x}'\Sigma\mathbf{x}),$$

with μ a vector, $\Sigma = (\sigma_{ij})_{1 \leq i,j \leq d}$ a symmetric positive-definite matrix and $\psi_d(t)$ a function called the characteristic generator.

They encompass a large number of distributions, see [29, Appendix]. Several properties have been developed in the case when the joint density exists, see [11, 2]. If it exists, the joint density $f(\mathbf{x}; \mu, \Sigma)$ has the following form:

$$f(\mathbf{x}; \mu, \Sigma) = c_d |\Sigma|^{-\frac{1}{2}} g_d\left(\frac{1}{2}(\mathbf{x} - \mu)'\Sigma^{-1}(\mathbf{x} - \mu)\right), \tag{4.4}$$

with $g_d(\cdot)$ a function called the density generator and c_d a normalising constant dependent of g_d (see [19]).

Elliptical distributions have been used in several areas including financial data analysis. Unlike the Gaussian distribution, the Student's t distribution has been known to account for fat tails and tail dependence.

4.2.1. The associated elliptically contoured copula

The copula of an elliptically contoured distribution is referred to as elliptically contoured copula or elliptical copula. This copula has been subject to numerous analysis, see for instance [7, 1, 5, 3]. One of the characteristics of elliptically contoured distributions is that their marginals $F_i(x)$ are also elliptically contoured with the same characteristic or density generator. If the d-dimensional copula density c exists the joint density f, the marginal densities f_i, the marginals F_i and the corresponding copula density satisfy the following relationship (see [7]):

$$f(x_1, ..., x_d) = c(F_1(x_1), ..., F_d(x_d)) \times \prod_{i=1}^{d} f_n(x_n).$$

Note that the process of standardising the marginal distributions of X uses strictly increasing transformations. As stated in proposition (3.3), copulas are invariant under such transformations. This implies that the copulas associated to $\mathbf{X} \sim El_d(\mu, \Sigma, \psi)$ are the same as the copulas associated to $\mathbf{X}^* \sim El_d(0, R, \psi)$. Here $R = (\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{iI}\sigma_{jJ}}})_{1 \le i,j \le d}$ is the corresponding "correlation" matrix implied by the positive-definite matrix $\Sigma = (\sigma_{ij})_{1 \le i,j \le d}$ (see [5, Theorem 5.2] or [7, 3]). Because of this, for our study of elliptical copulas we assume $\mathbf{X} \sim El_d(R, \psi)$ with $R = (\rho_{ij})_{1 \le i,j \le d}$, which covers the more general case $\mathbf{X} \sim El_d(\mu, \Sigma, \psi)$.

Equation (3.4) implies that the associated copulas of $\mathbf{X}$ are determined by the joint distribution and the inverse of the marginal distributions. In general, there is no closed-form expression for the elliptical copula but it can be expressed in terms of multidimensional integrals of the joint density $f(\mathbf{x};R)$. This case covers a wide variety of distributions, see e.g. [29, Appendix]. In the following proposition we prove an identity for the associated copulas for this general case.

Proposition 4.2. *Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with multivariate elliptical distribution of Definition 4.1, with correlation matrix $R = (\rho_{ij})_{1 \le i,j \le d}$, that is $\mathbf{X} \sim El_d(R, \psi)$ and let $\mathbf{D}$ be a type of dependence. Then the associated $\mathbf{D}$-copula of $\mathbf{X}$ is the same as the distributional copula of $\mathbf{X}^+ \sim El_d(\wp R \wp, \psi)$, with $\wp$ a diagonal matrix (all values in it are zero except for the values in its diagonal) $\wp \in M_{d \times d}$, whose diagonal is $\mathbf{p} = (p_1, ..., p_d)$ with*

$$p_i = \begin{cases} 1 & \text{if } D_i = L \\ -1 & \text{if } D_i = U \end{cases},$$

for $i \in \{1, ..., d\}$.

Proof: The vector $\wp\mathbf{X}$ is equal to $(T_1(X_1), ..., T_d(X_d))$ with $T_i(x) = p_i x$, $i \in \{1, ..., d\}$. Using Proposition (3.4), the distributional copula of $\wp\mathbf{X}$ is the associated $\mathbf{D}$-copula of $\mathbf{X}$. From the stochastic representation of $\mathbf{X}$ (see [8]), it follows that $\wp\mathbf{X} \sim El_d(\wp R \wp', \psi)$ (see [5, Theorem 5.2]). □

The symmetric nature of the elliptically contoured distributions and copula is well known. It follows from proposition (4.2) that elliptical copulas are complement symmetric.

Corollary 4.1. *Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector with multivariate elliptical distribution of Definition 4.1, $\mathbf{X} \sim El_d(R, \psi)$. Then $\mathbf{X}$ is complement symmetric according to Definition 3.5.*

Proof: Let $\mathbf{D}$ be a type of dependence and $\mathbf{D}^{\complement}$ the complement dependence of Definition 3.5. Denote $\wp_{\mathbf{D}}$ and $\wp_{\mathbf{D}^{\complement}}$ the corresponding diagonal matrices of proposition (4.2).

It is clear that $\wp_{\mathbf{D}^{\complement}} = -\wp_{\mathbf{D}}$, which implies

$$\wp_{\mathbf{D}^{\complement}} \cdot R \cdot \wp_{\mathbf{D}^{\complement}} = \wp_{\mathbf{D}} \cdot R \cdot \wp_{\mathbf{D}}.$$

Hence, both $C_{\mathbf{D}}$ and $C_{\mathbf{D}^{\complement}}$ are equal to the distributional copula of $\mathbf{X}^+ \sim El_d(\wp_{\mathbf{D}} R \wp_{\mathbf{D}}, \psi)$. □

Proposition (4.2) makes it possible to use the results regarding elliptical copulas in associated copulas. This also includes the analysis of tail dependence. In the bivariate case Klüppelberg et al. [18] and Schmidt [25] studied positive tail dependence in elliptical copulas under regular variation conditions. The Gaussian copula does not account for positive tail dependence, proposition (4.2) implies that it does not account for tail dependence for all $\mathbf{D}$. In contrast the Student's t copula does account for tail dependence (see [15, 23, 3]). We now analyse this copula into more detail.

4.2.2. The multivariate student's t associated tail dependence function

The Student's t copula is well known for accounting for stylised facts such as fat tail and the presence of tail dependence (see [15, 23, 3]). The Student's t copula with ν degrees of freedom and correlation matrix R is expressed in terms of integrals of its corresponding density $t_{\nu,R}$.

$$C(\mathbf{u}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^d |R|}} \left(1 + \frac{\mathbf{x}' R^{-1} \mathbf{x}}{\nu}\right)^{-\frac{\nu+d}{2}} d\mathbf{x},$$

with $\mathbf{u} = (u_1, ..., u_d)$ and $\mathbf{x} = (x_1, ..., x_d)'$.

Unlike the multivariate Gaussian distribution, the case of $R = I$ does not correspond to the independence case, see [12]. It must also be noted that the perfect dependence cases are not covered by this copula. That is, the cases when $R = (\rho_{ij})_{1\leq i,j\leq d}$ satisfies $\rho_{ij} = 1$ if $i, j \in S_1$ or $i, j \in S_2$, and $\rho_{ij} = -1$ if $i \in S_1,\ j \in S_2$ or $i \in S_2,\ j \in S_1$, with S_1 and S_2 disjoint sets that satisfy $S_1 \cup S_2 = \{1, ..., d\}$. In this case $R = (\rho_{ij})_{1\leq i,j\leq d}$ can be expressed as

$$R = \wp \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \wp$$

with $\wp \in M_{d\times d}$ a diagonal matrix, whose values in the diagonal are 1 if $i \in S_1$ and -1 if $i \in S_2$, $i \in \{1, ..., d\}$. Inductively on d, it is straightforward to prove that the determinant of a matrix of ones is zero. It follows that $|R| = 0$, and the copula is not defined in this case. The Student's t copula is known for accounting for several types of tail dependence. McNeil et al. [20] proved that, in the bivariate case, regardless of the value of the correlation coefficient ρ, the lower and upper tail dependence coefficients are positive. Nikoloulopoulos et al. [23] analysed in detail the extreme value properties of this copula and obtained an expression for the lower and upper tail dependence functions among other results. More recently, in the bivariate case, Joe [15], obtained an expression for the $\mathbf{D} = (L, U)$ and the $\mathbf{D} = (U, L)$ tail dependence coefficients proving that this copula accounts for negative tail dependence. In this subsection we present the expression for the associated $\mathbf{D}$-tail dependence function of the multivariate Student's t copula. Given that this function is positive for $|R| \neq 0$ and for all $\mathbf{D}$, the Student's t copula accounts for all types of tail dependence. This result follows from [23, Theorem 2.3] and proposition (4.2).

Proposition 4.3. *Let $\mathbf{X} = (X_1, ..., X_d)$ have multivariate t distribution with ν degrees of freedom, and correlation matrix $R = (\rho_{ij})_{1\leq i,j\leq d}$, that is $\mathbf{X} \sim T_{d,\nu,R}$. Let $\mathbf{D} = (D_1, ..., D_d)$ be a type of dependence. Then the associated $\mathbf{D}$-tail dependence function $b_{\mathbf{D}}$ is given by*

$$b_{\mathbf{D}}(w) = \sum_{j=1}^{d} w_j T_{d-1,\nu+1,R'_j} \left(\sqrt{\frac{\nu+1}{1-\rho_{ij}^2}} \left[-\left(\frac{w_i}{w_j}\right)^{-\frac{1}{\nu}} + p_i p_j \rho_{ij} \right], i \in I_j \right),$$

with

$$R_j^* = \begin{pmatrix} 1 & \cdots & \rho^*_{1,j-1;j} & \rho^*_{1,j+1;j} & \cdots & \rho^*_{1,d;j} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \rho^*_{j-1,1;j} & \cdots & 1 & \rho^*_{j-1,j+1;j} & \cdots & \rho^*_{j-1,d;j} \\ \rho^*_{j+1,1;j} & \cdots & \rho^*_{j+1,j-1;j} & 1 & \cdots & \rho^*_{j+1,j-1;j} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^*_{d,1;j} & \cdots & \rho^*_{d,j-1;j} & \rho^*_{d,j+1;j} & \cdots & 1 \end{pmatrix};$$

$\rho^*_{i,k;j} = p_i p_k \frac{\rho_{ik}-\rho_{ij}\rho_{kj}}{\sqrt{1-\rho_{ij}^2}\sqrt{1-\rho_{kj}^2}}$, *the modified partial correlations;* $I_j = I - \{j\}$ *and*

$$p_j = \begin{cases} 1 & \text{if } D_j = L \\ -1 & \text{if } D_j = U \end{cases},$$

for $j \in \{1, ..., d\}$.

Proof: The definition presented in this work for the tail dependence functions has the same form for different dependencies. The only difference is the underlying associated copula. Proposition (4.2) then implies that the associated **D**-tail dependence function of the random vector $\mathbf{X} \sim T_{d,\nu,R}$ is the lower tail dependence function of the vector $\mathbf{X}^+ \sim T_{d,\nu,\wp R \wp}$. $\wp$ is the diagonal matrix, whose diagonal is $\mathbf{p} = (p_1, ..., p_d)$ with

$$p_i = \begin{cases} 1 & \text{if } D_i = L \\ -1 & \text{if } D_i = U \end{cases}$$

for $i \in \{1, ..., d\}$.

The modified correlation matrix is $\wp R \wp = R^* = (\rho^*_{ij})_{1 \le i,j \le d}$, it follows that

$$(\rho^*_{ij})_{1 \le i,j \le d} = (p_i p_j \rho_{ij})_{1 \le i,j \le d}.$$

Hence $(\rho^*_{ij})^2 = p_i^2 p_j^2 \rho_{ij} = 1 \cdot 1 \cdot \rho_{ij}^2 = \rho_{ij}^2$. Under this change, the partial correlations are modified as follows

$$\rho^*_{i,k;j} = p_i p_k \frac{\rho_{ik} - \rho_{ij}\rho_{kj}}{\sqrt{1-\rho_{ij}^2}\sqrt{1-\rho_{kj}^2}}.$$

The result is then implied by [23, Theorem 2.3]. □

This proposition implies that the Student's t copula accounts for all 2^d dependencies simultaneously. This includes simultaneous positive and non-positive tail dependence. The level of tail dependence in each orthant is determined by the correlation matrix R. It is interesting to note that positive correlation can occur conteporaneously with non-positive tail dependence. This comes from the fact that correlation and the tail dependence refer to different features of the dependence structure between variables. For example, in the bivariate case, high negative correlation implies high LU and UL tail dependence but does not rule out lower and upper tail dependence. In that case, the variables might generally exhibit negative depenence, but when it comes to extreme values they can also be positively dependent. Although simultaneous positive and non-positive tail dependence and symmetry can be present in empirical data, these assumptions must be verified before considering this model.

5. CONCLUSIONS AND FUTURE WORK

In this paper we introduce concepts to analyse, in the multivariate case, the whole dependence structure among random variables. We consider the 2^d different orthants of dimension d. For this, we present the $\mathbf{D}$-probability functions, associated copulas, associated tail dependence functions and associated TDCs. These concepts are meant to measure non-positive dependence and tail dependence, complementing the use of the distributional and survival copulas C and $\widehat{C}$ and the lower and upper tail dependence functions and TDCs. These concepts can be used to analyse non-positive dependence and tail dependence of copula modelsby considering all the orthants. This work is divided into two main parts, in the first one we define and study the concepts to analyse general dependence and in the second one we study the whole dependence structure of two copula families.

We begin the first part by defining the $\mathbf{D}$-probability functions to analyse probabilities in the different orthants. We then present a version of Sklar's theorem that links $\mathbf{D}$-probability functions with the associated copulas. Together with the distributional and the survival copulas, the other associated copulas characterise the dependence structure among random variables. With them it is possible to analyse all types of dependence to cover the whole dependence structure among random variables. We study the associated copulas of a random vector and present an expression for the relationship among all of these copulas. After this, we prove properties of associated copulas regarding symmetry and exchangeability. We then prove that they are invariant under strictly increasing transformations and characterise the copula of a vector after using monotone transformations. For the last part of this analysis, we introduce the associated tail dependence functions and associated tail dependence coefficients of a random vector. With them we can analyse its tail dependence in the different orthants.

For the second part we use the concepts and results obtained in the first part of the paper to analyse two examples of copula models. The first example corresponds to the perfect dependence models. The corresponding copulas are a generalisation of the Fréchet copula bounds of the bivariate case, they correspond to the use of strictly monotone transformations on a random variable. Accordingly, we name these copulas the monotonic copulas. The second example corresponds to the elliptical contoured distributions. For this example, we also obtain an expression for the corresponding associated copulas. As expected the Gaussian copula does not account for any type of tail dependence, regardless of the correlation matrix. We present an expression for the associated tail dependence function of the Student's t copula. This result proves that this copula model accounts for tail dependence in all orthants. The Student's t copula has proven to be a better copula model than the Gaussian copula when modelling empirical data such as financial. It is well known that this data has heavy tails and extreme dependencies and the assumption of only positive tail dependence has proven to be unrealistic. It is not surprising, but yet interesting that the

Student's t copula accounts for extreme dependencies of all types simultaneously. Non-positive tail dependence can be desirable in the context of hedging strategies. This tail dependence can minimise the risks and variability of a portfolio in times of economic crisis when extreme values are likely to appear.

The results obtained in this work aim to help in the understanding of the dependence structure of a multivariate random vector. With them it is also possible to analyse the dependence structure implied by different copula models. Without analysing general dependence, the analysis in these models is therefore incomplete. There are several areas where future research regarding general dependence is worth being pursued. For instance, it must be noted that the use of **D**-probability functions is not restricted to copula theory. The analysis of probabilities in the multivariate case has sometimes been centred in distribution functions, but, just like survival functions, **D**-probability functions can serve different purposes in dependence analysis. Another possibility is the use of nonparametric estimators to measure non-positive tail dependence, as the use of these estimators has been restricted to the lower and upper cases. Further to this, with the formulas presented for the associated copulas, it is possible to extend the analysis to other copula models with closed-form expressions. This includes copulas such as the Archimedean and other models based on Laplace transforms. Other interesting examples of copula models are the vine copulas, the use of these copulas has proven to provide a flexible approach to tail dependence and account for asymmetric positive tail dependence (see for instance [24, 16]).

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REFERENCES

[1] Abdous, B., Genest, C. and Rémillard, B. (2005). *Dependence properties of meta-elliptical distributions*. In "Statistical Modelling and Analysis for Complex Data Problems" (P. Duchesne. and B. Rémillard, Eds.), Kluwer, Dordrecht, 1–15.

[2] Das Gupta, S., Eaton, M. L., Olkin, I., Perlman, M., Savage, L. J. and Sobel, M. (1972). *Inequalities on the probability content of convex regions for elliptically contoured distributions*. In "Proceedings of the Sixth Berkeley Sympo-

sium on Mathematical Statistics and Probability, 2" (L. Le Cam, J. Neyman abd E. Scott, Eds.), University of California Press, Berkeley, 241–264.

[3] DEMARTA, S. AND MCNEIL, A. J. (2005). The t copula and related copulas, *International Statistical Review*, **73**, 1, 111–129.

[4] EMBRECHTS, P., LAMBRIGGER, D. AND WUTHRICH, M. V. (2009). Multivariate extremes and aggregation of dependent risks: examples and counter-examples, *Extremes*, **12**, 1, 107–127.

[5] EMBRECHTS, P., LINDSKOG, F. AND MCNEIL, A. (2001). *Modelling dependencies with copulas and applications to risk management*. In "Handbook of Heavy Tailed Distributions in Finance, Chapter 8" (S.T. Rachev, Ed.), Elsevier, North Holland, 329–384.

[6] EMBRECHTS, P., MCNEIL, A. J. AND STRAUMANN, D. (2002). *Correlation and dependency in risk management: properties and pitfalls*. In "Risk Management: Value at Risk and Beyond" (M. Dempster, Ed.), Cambridge University Press, Cambridge, 176–223.

[7] FANG, H. B., FANG, K. T. AND KOTZ, S. (2002). The meta-elliptical distributions with given marginals, *Journal of Multivariate Analysis*, **82**, 1, 1–16.

[8] FANG, K. T., KOTZ, S. AND NG, K. W. (1990). *Symmetric Multivariate and Related Distributions*, Chapman and Hall, London.

[9] FRÉCHET, M. (1951). Sur les tableaux de corrélation dont les marges sont données, *Annales de l'Université de Lyon, Sciences Mathématiques et Astronomié*, **3**, 14, 53–77.

[10] GEORGES, P., LAMY, A. G., NICOLAS, E., QUIBEL, G. AND RONCALLI, T. (2001). Multivariate survival modelling: a unified approach with copulas, *Groupe de Recherche Opérationnelle Crédit Lyonnais France, Unpublished results.*

[11] GUPTA, A. K. AND VARGA, T. (1993). *Elliptically Contoured Models in Statistics*, Kluwer Academic Publishers, Netherlands.

[12] HULT, H. AND LINDSKOG, F. (2002). Multivariate extremes, aggregation and dependence in elliptical distributions, *Advances in Applied Probability*, **34**, 3, 587–608.

[13] JOE, H. (1993). Parametric families of multivariate distributions with given margins, *Journal of Multivariate Analysis*, **46**, 2, 262–282.

[14] JOE, H. (1997). *Multivariate Models and Dependence Concepts*, Chapman & Hall, London.

[15] JOE, H. (2011). *Tail dependence in vine copulae*. In "Dependence Modeling: Vine Copula Handbook, Chapter 8" (D. Kurowicka and H. Joe, Eds.), World Scientific, Singapore, 165–189.

[16] JOE, H., LI, H. AND NIKOLOULOPOULOS, A. K. (2010). Tail dependence functions and vine copulas, *Journal of Multivariate Analysis*, **101**, 1, 252–270.

[17] KELKER, D. (1970). Distribution theory of spherical distributions and location-scale parameter generalization, *Sankhya: The Indian Journal of Statistics, Series A*, **32**, 4, 419–430.

[18] KLÜPPELBERG, C., KUHN, G. AND PENG, L. (2008). Semi-parametric models for the multivariate tail dependence function - the asymptotically dependent case, *Scandinavian Journal of Statistics*, **35**, 4, 701–718.

[19] Landsman, Z. and Valdez, E. A. (2003). Tail conditional expectations for elliptical distributions, *North American Actuarial Journal*, **7**, 4, 55–71.

[20] McNeil, A., Frey, R. and Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton Series in Finance, New Jersey.

[21] Mikosch, T. (2006). Copulas: tales and facts, *Extremes*, **9**, 1, 3–20.

[22] Nelsen, R. B. (2006). *An Introduction to Copulas*, 2nd edn. Springer, New York.

[23] Nikoloulopoulos, A. K., Joe, H. and Li, H. (2009). Extreme value properties of multivariate t copulas, *Extremes*, **12**, 2, 129–148.

[24] Nikoloulopoulos, A. K., Joe, H. and Li, H. (2012). Vine copulas with asymmetric tail dependence and applications to financial return data, *Computional Statistics and Data Analysis*, **56**, 11, 3659–3673.

[25] Schmidt, R. (2002). Tail dependence for elliptically contoured distributions, *Mathematical Methods of Operations Research*, **55**, 2, 301–327.

[26] Schmidt, R. and Stadtmüller, U. (2006). Non-parametric estimation of tail dependence, *Scandinavian Journal of Statistics*, **33**, 2, 307–335.

[27] Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs margès, *Publications de l'Institut de Statistique de L'Universitè de Paris*, **8**, 1, 229–231.

[28] Taylor, M. D. (2007). Multivariate measures of concordance, *Annals of the Institute of Statistical Mathematics*, **59**, 4, 789–806.

[29] Valdez, E. A. and Chernih, A. (2003). Wang's capital allocation formula for elliptically-contoured distributions, *Insurance: Mathematics and Economics*, **33**, 3, 517–532.

[30] Zhang, M. H. (2007). Modelling total tail dependence along diagonals, *Insurance: Mathematics and Economics*, **42**, 1, 77–80.